Second Edition

Digital Control of Dynamic Systems

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CHAPTER 1

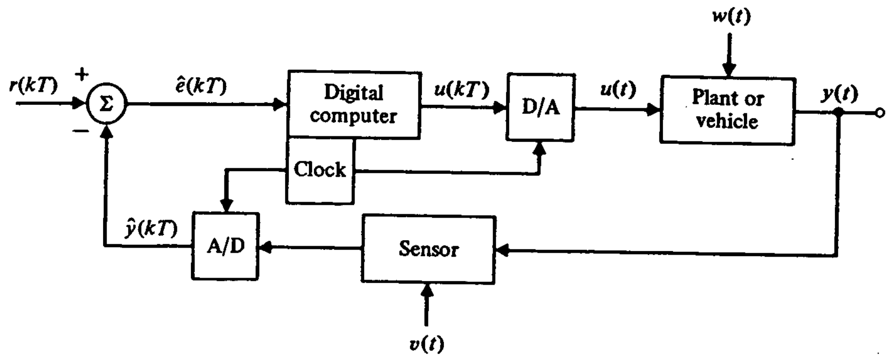
Introduction

1.1 PROBLEM DEFINITION

The control of physical systems with a digital computer is becoming more and more common. Aircraft autopilots, mass-transit vehicles, oil refineries, paper-making machines, and countless electromechanical servomechanisms are among the many existing examples. Furthermore, many new digital control applications are being stimulated by microprocessor technology including control of various aspects of automobiles and household appliances. Among the advantages of digital logic for control are the increased flexibility of the control programs and the decision-making or logic capability of digital systems, which can be combined with the dynamic control function to meet other system requirements.

The digital controls studied in this book are for closed-loop (feedback) systems in which the dynamic response of the process being controlled is a major consideration in the design. A typical topology of the elementary type of system that will occupy most of our attention is sketched schematically in Fig. 1.1. This figure will help to define our basic notation and to introduce several features that distinguish digital controls from those implemented with analog devices. The process to be controlled is called the plant and may be any of the physical processes mentioned above whose satisfactory response requires control action.

By "satisfactory response" we mean that the plant output, $y(t)$, is to be forced to follow or track the reference input, $r(t)$, despite the presence of disturbance inputs to the plant [$w(t)$ in Fig. 1.1] and despite errors in the sensor [represented by $v(t)$ in Fig. 1.1]. It is also essential that the tracking succeed even if the dynamics of the plant should change somewhat during



Notation:
r = reference or command inputs
u = control or actuator input signal
y = controlled or output signal
$\hat{y}$ = instrument or sensor output, usually an approximation to or estimate of y. (For any variable, say θ, the notation $\hat{\theta}$ is now commonly taken from statistics to mean an estimate of θ.)
$\hat{e}$ = $r - \hat{y}$ = indicated error
e = $r - y$ = system error
w = disturbance input to the plant
v = disturbance or noise in the sensor
A/D = analog-to-digital converter
D/A = digital-to-analog converter

Figure 1.1 Block diagram of a basic control system.

the operation. The process of holding $y(t)$ close to $r(t)$, including the case where $r \equiv 0$, is referred to generally as the process of *regulation*. A system that has good regulation in the presence of disturbance signals is said to have good *disturbance rejection*. A system that has good regulation in the face of changes in the plant parameters is said to have low *sensitivity* to these parameters. A system that has both good disturbance rejection and low sensitivity we call *robust*.

The means by which robust regulation is to be accomplished is through the control inputs to the plant [$u(t)$ in Fig. 1.1]. It was discovered long ago[1] that a scheme of feedback wherein the plant output is measured (or sensed) and compared directly with the reference input has many advantages in the effort to design robust controls over systems that do not use such feedback. Much of our effort in later parts of this book will be devoted to illustrating this discovery and demonstrating how to exploit the advantages of feedback. However, the problem of control as discussed thus far is in no way restricted

[1]See especially the book by Bode (1945).

to digital control. For that we must consider the unique features of Fig. 1.1 introduced by the use of a digital device to generate the control action.

We consider first the action of the analog-to-digital (A/D) converter on a signal. This device acts on a physical variable, most commonly an electrical voltage, and converts it into a stream of numbers. In Fig. 1.1, the A/D converter acts on the sensor output and supplies numbers to the digital computer. It is common for the sensor output, $\hat{y}$, to be sampled and to have the error formed in the computer. We need to know the times at which these numbers arrive if we are to analyze the dynamics of this system.

In this book we will make the assumption that all the numbers arrive with the same fixed period T, called the *sampling period*. In practice, digital control systems sometimes have varying sample periods and/or different periods in different feedback paths. Usually there is a clock as part of the computer logic which supplies a pulse or *interrupt* every T seconds, and the A/D converter sends a number to the computer each time the interrupt arrives. An alternative implementation is simply to access the A/D upon completion of each cycle of the code execution, a scheme often referred to as *free running*. In the first case the sample period is precisely fixed; in the latter case the sample period is essentially fixed by the length of the code, providing no logic branches are present that could vary the amount of code executed. Thus in Fig. 1.1 we identify the sequence of numbers into the computer as $\hat{e}(kT)$. We conclude from the periodic sampling action of the A/D converter that some of the signals in the digital control system, like $\hat{e}(kT)$, are variable only at discrete times. We call these variables *discrete signals* to distinguish them from variables like w and y, which change continuously in time. A system having both discrete and continuous signals is called a *sampled-data* system.

In addition to generating a discrete signal, however, the A/D converter also provides a *quantized* signal. By this we mean that the output of the A/D converter must be stored in digital logic composed of a finite number of digits. Most commonly, of course, the logic is based on binary digits (i.e., bits) composed of 0's and 1's, but the essential feature is that the representation has a finite number of digits. A common situation is that the conversion of y to $\hat{y}$ is done so that $\hat{y}$ can be thought of as a number with a fixed number of places of accuracy. If we plot the values of y versus the resulting values of $\hat{y}$ we can obtain a plot like that shown in Fig. 1.2. We would say that $\hat{y}$ has been truncated to one decimal place, or that $\hat{y}$ is *quantized* with a q of 0.1, since $\hat{y}$ changes only in fixed quanta of, in this case, 0.1 units. (We will use q for quantum size, in general.) Note that quantization is a nonlinear function. A signal that is both discrete and quantized is called

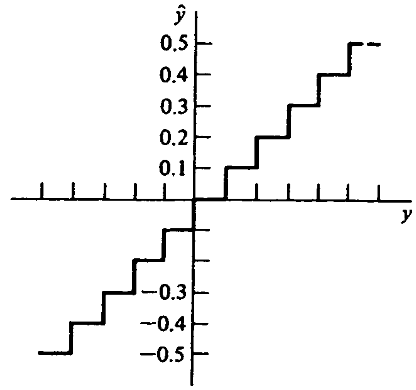


Figure 1.2 Plot of output versus input characteristics of the A/D converter.

a *digital signal.* Not surprisingly, digital computers in this book process digital signals.

In a real sense the problems of analysis and design of *digital controls* are concerned with taking account of the effects of the sampling period T and the quantization size q. If both T and q are extremely small (sampling frequency 50 or more times the system bandwidth with a 16-bit word size), digital signals are nearly continuous, and continuous methods of analysis and design can be used. The resulting design could be converted to the digital format for implementation by using the *emulation* method described in Chapter 5. We will be interested in this text in gaining an understanding of the effects of all sample rates, fast and slow, and the effects of quantization for large and small word sizes. Many systems are originally conceived with fast sample rates, and the computer is specified and frozen early in the design cycle; however, as the designs evolve, more demands are placed on the system, and the only way to accommodate the increased computer load is to slow down the sample rate. Furthermore, for cost-sensitive digital systems, the best design is the one with the lowest cost computer that will do the required job. That translates into being the computer with the slowest speed and the smallest word size. We will, however, treat the problems of varying T and q separately. We first consider q to be zero and study discrete and sampled-data (combined discrete and continuous) systems that are linear. In Chapter 7 we will analyze in more detail the source and the effects of quantization, and we will discuss in Chapters 5 and 10 specific effects of sample-rate selection.

It is worthy to note that *the single most important* impact of implementing a control system digitally is the delay associated with the D/A

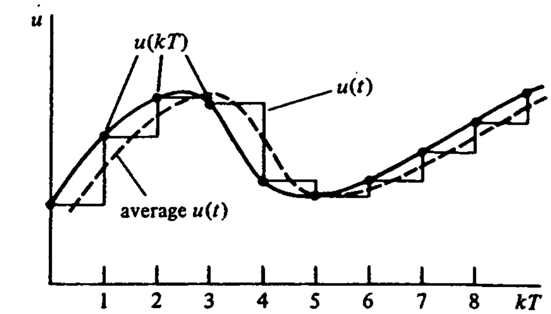


Figure 1.3 The delay due to the hold operation.

converter. Each value of $u(kT)$ in Fig. 1.1 is typically held constant[2] until the next value is available from the computer. Thus the continuous value of $u(t)$ consists of steps (see Fig. 1.3) that, on the average, lag $u(kT)$ by $T/2$, as shown by the dashed line in the figure. If one simply incorporates this $T/2$ delay in a continuous analysis of a digital system, excellent agreement results for many reasonable sample rates. This point will be explained further in Chapters 3 and 5.

Our approach to the design of digital controls is to assume a background in continuous systems and to relate the comparable digital problem to its continuous counterpart. We will develop the essential results, from the beginning, in the domain of discrete systems, but we will call upon previous experience in continuous-system analysis and in design to give alternative viewpoints and deeper understanding of the results. In order to make meaningful these references to a background in continuous-system design, we will review the concepts and define our notation as required.

1.2 EXAMPLE SYSTEMS FOR STUDY

In order to guide the discussion in the following chapters we have developed models for six example control problems in Appendix A. The first of these is a satellite attitude control problem in which the plant transfer function is the double integrator

$$G_1(s) = 1/s^2. \tag{1.1}$$

This example is simple, but with two poles on the stability boundary, this transfer function must be controlled with care. The second example is a

[2]Called a *Zero Order Hold* or ZOH.

servomechanism, motivated by a radar tracking antenna. The basic control dynamics of the amplifier–motor–load system is second order and has a transfer function with one pole at the origin and one pole on the negative real axis. The transfer function can be normalized to be

$$G_2(s) = 1/s[(s+1)]. \tag{1.2}$$

The third example to be used to illustrate digital control comes from the process industries—a mixing process. The normalized transfer function that represents this system is

$$G_3(s) = e^{-T_d s}/(s+1). \tag{1.3}$$

The obvious feature of this model is the delay term, $e^{-T_d s}$, which is called a *transportation delay*. Such delays are very difficult to control, and the example affords us an opportunity to study these difficulties in the context of digital control. Also, because there is no internal integrator in this model, good steady-state accuracy requires integral control, which presents an interesting but common design problem.

In many electromechanical systems, there is a flexible mechanical member *between* the actuator and the position sensor. The effect of this flexibility is to introduce a complex pole into the transfer function that often has very low damping and creates difficulties in providing a fast, stable system. This situation arises frequently enough to warrant a name: the *noncolocated* sensor and actuator. An example is the control of a robot arm where the end point of the arm is measured directly. The normalized transfer function for this kind of system is

$$G_4(s) = \frac{1}{s^2(s^2/\omega_p^2 + 2\zeta_p s/\omega_p + 1)}. \tag{1.4}$$

A variation of this problem occurs if the sensor is on the same mass as the actuator, the colocated case. This situation can occur in the positioning of a large radio telescope or in the control of a satellite with flexible solar panels. In this case the transfer function includes complex zeros with almost no damping in addition to the lightly damped complex poles. The normalized transfer function is an extension of (1.4) and is given by

$$G_5(s) = \frac{s^2/\omega_z^2 + 2\zeta_z s/\omega_z + 1}{s^2(s^2/\omega_p^2 + 2\zeta s/\omega_p + 1)}. \tag{1.5}$$

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Our sixth example illustrates multivariable control. Its key feature is that it has two controls and two outputs, so the transfer function is a matrix. It comes from the process industry and illustrates control of the liquid level and total pressure (head) in the head box of a paper machine. State space methods are the most convenient to describe this example; however, it can also be described by the transfer function

$$G_6(s) = \frac{1}{\Delta(s)} \begin{bmatrix} a(s) & b(s) \\ c(s) & d(s) \end{bmatrix}, \tag{1.6}$$

where $\Delta(s)$ is the system characteristic equation.

A seventh example, used in a multivariable design in Chapter 9, is a magnetic tape drive where the tape is driven at both ends and the outputs of interest are the tape position and tension. It also can be described by (1.6).

1.3 OVERVIEW OF DESIGN APPROACH

An overview of the path we plan to take toward the design of digital controls will be useful before we begin the specific details. As mentioned above, we place systems of interest in three categories according to the nature of the signals present. These are discrete systems, sampled-data systems, and digital systems.

In discrete systems all signals vary at discrete times only. We will begin with an analysis of these in Chapter 2 and develop the z-transform of discrete signals and "pulse"-transfer functions for linear constant discrete systems. We also develop discrete transfer functions of continuous systems that are sampled, systems that are called sampled-data systems. We develop the equations and give examples using both transform methods and state-space descriptions. Having the discrete transfer functions, we consider the issue of the dynamic response of discrete systems.

A sampled-data system has both discrete and continuous signals, and often it is important to be able to compute the continuous time response. For example, with a slow sampling rate, there can be significant *ripple* between sample instants. Such situations are studied in Chapter 3. Here we are concerned with the question of data extrapolation to convert discrete signals as they might emerge from a digital computer into the continuous signals necessary for providing the input to one of the plants described above. This action typically occurs in conjunction with the D/A conversion. In addition to data extrapolation, we consider the analysis of sampled signals from the

viewpoint of continuous analysis. For this purpose we introduce impulse modulation as a model of sampling, and we use Fourier analysis to give a clear picture for the ambiguity that can arise between continuous and discrete signals, also known as *aliasing*. The plain fact is that more than one continuous signal can result in exactly the same sample values. If a sinusoidal signal, y_1, at frequency f_1 has the same samples as a sinusoid y_2 of a *different* frequency f_2, y_1 is said to be an *alias* of y_2. A corollary of aliasing is the *sampling theorem*, which specifies the conditions necessary if this ambiguity is to be removed and only one continuous signal allowed to correspond to a given set of samples.

As a special case of discrete systems and as the basis for the emulation design method, we consider discrete equivalents to continuous systems, which is one aspect of the field of *digital filters*. Digital filters are discrete systems designed to process discrete signals in such a fashion that the digital device (a digital computer, for example) can be used to replace a continuous filter. Our treatment in Chapter 4 will concentrate on the use of discrete filtering techniques to find discrete equivalents of continuous-control compensator transfer functions. Again, both transform methods and state-space methods are developed to help understanding and computation of particular cases of interest.

Once we have developed the tools of analysis for discrete and sampled systems we can begin the design of feedback controls. Here we divide our techniques into two categories: transform[3] and state-space[4] methods. In Chapter 5 we study the transform methods of the root locus and the frequency response as they can be used to design digital control systems. The use of state-space techniques for design is introduced in Chapter 6. For purposes of understanding the design method, we rely mainly on "pole assignment," a scheme for forcing the closed-loop poles to be in desirable locations. We discuss the selection of the desired pole locations and point out the advantages of using the optimal control methods covered in Chapter 9. Chapter 6 includes control design using feedback of all the "state variables" as well as methods for estimating the state variables that do not have sensors directly on them. A study of quantization effects in Chapter 7 both presents a "worse-case" analysis and introduces the idea of random

[3]Named because they use the Laplace or Fourier transform to represent systems.

[4]The state space is an extension of the space of displacement and velocity used in physics. Much that is called *modern control theory* uses differential equations in state-space form. We introduce this representation in Chapter 2 and use it extensively afterwards, especially in Chapters 6 and 9.

signals in order to describe a method for treating the "average" effects of this important nonlinearity.

The last five chapters cover more advanced topics that are essential for most complete designs. The first of these topics is *identification*, introduced in Chapter 8. Here the matter of model making is extended to the use of experimental data to verify and correct a theoretical model or to supply a dynamic description based only on input–output data. Only the most elementary of the concepts in this enormous field can be covered, of course. We present the method of least squares and some of the concepts of maximum likelihood. In Chapter 9 the topic of optimal control is introduced, with emphasis on the steady-state solution for linear constant discrete systems with quadratic loss functions. The results are a valuable part of the designer's repertoire and are the only techniques presented here suitable for handling multivariable designs. Chapter 10 presents a topic with specific application to sampled-data and digital controls: the question of sampling-rate selection. In our earlier analysis we develop methods for examining the effects of different sampling rates, but in this chapter we consider for the first time the question of sample rate as a design parameter.

In Chapter 11, an introduction to the most important issues and techniques for the analysis and design of nonlinear sampled-data systems is given. The analysis methods treated are the describing function, equivalent linearization, and Lyapunov's second method of stability analysis. Design techniques described are the use of inverse nonlinearity, optimal control (especially time-optimal control), and adaptive control. In Chapter 12, many generic issues concerned with the practical application of digital controls are described. The chapter includes a case study of a disk-drive design, and treatment of both implementation and manufacturing issues are discussed.

1.4 COMPUTER-AIDED DESIGN

As with any engineering design method, design of control systems requires many computations that are greatly facilitated by a good library of well-documented computer programs. In designing practical digital control systems, and especially in iterating through the methods many times to meet essential specifications, an interactive computer-aided design (CAD) package with simple access to plotting graphics is crucial. Many commercial control system CAD packages are available which satisfy that need, and much of the discussion in the book assumes that a designer has access to one of them. Specific CAD routines that can be used for performing calculations for a

topic in the book are indicated in the text, and the routine names for some of the popular CAD packages are listed in Table E.1 in Appendix E.

CAD support for a designer is universal; however, it is essential that the designer is able to work out very simple problems by hand in order to have some idea about the reasonableness of the computer's answers.

1.5 SUMMARY

In this chapter we introduced the subject of digital control design and defined the variables that characterize discrete, sampled-data, and digital control systems. Pointing out that we will be mainly concerned with simple single-loop linear constant systems, we gave the (highly simplified) transfer functions of six physical examples, which will be used to illustrate the analysis and design techniques as they are presented in the book. These examples represent aspects of a space vehicle, a servomechanism, temperature control, a flexible structure, a paper machine and a magnetic tape drive.

Finally, we gave an overview of our study, from the analysis of discrete and sampled-data systems to the design of digital controls via transform and state-space techniques. We pointed out that, following development of design methods for the most elementary case, we will consider the topics of quantization, identification, linear optimal control, selection of sampling rate, nonlinear controls, and issues in the implementation of digital control.

1.6 SUGGESTIONS FOR FURTHER READING

Several histories of feedback control are readily available, including a *Scientific American Book* (1955), and the study of Mayr (1970). A good discussion of the historical developments of control is given by Dorf (1980) and by Fortmann and Hitz (1977), and many other references are cited by these authors for the interested reader. One of the earliest published studies of control systems operating on discrete time data (sampled-data systems in our terminology) is given by Hurewicz in Chapter 5 of the book by James, Nichols, and Phillips (1947).

The ideas of tracking and robustness embody many elements of the objectives of control system design. The concept of tracking contains the requirements of system stability, good transient response, and good steady-state accuracy, all concepts fundamental to every control system. Robustness is a property essential to good performance in practical designs because real

parameters are subject to change and because external, unwanted signals invade every system. Discussion of performance specifications of control systems is given in most books on introductory control, including Franklin, Powell, and Emami-Naeini (1986). We will study these matters in later chapters with particular reference to digital control design.

A comprehensive treatment of conversion techniques between analog and digital signals is given by Hnatek (1976).

A comprehensive text concerned with writing equations of motion for systems in a form suitable for control studies is Cannon (1967).

PROBLEMS AND EXERCISES

1.1 Suppose a radar search antenna at San Francisco airport rotates at 6 rev/min, and data points corresponding to the position of flight 1081 are plotted on the controller's screen once per antenna revolution. Flight 1081 is traveling directly toward the airport at 540 mi/hr. A feedback control system is established through the controller who gives course corrections to the pilot. He wishes to do so each 9 mi of travel of the aircraft, and his instructions consist of course headings in integral degree values.

- **a)** What is the sampling rate, in seconds, of the range signal plotted on the radar screen?
- **b)** What is the sampling rate in seconds, of the controller's instructions?
- **c)** Identify the following signals as continuous, discrete, or digital:
  - **i)** the aircraft's range from the airport,
  - **ii)** the range data as plotted on the radar screen,
  - **iii)** the controller's instructions to the pilot,
  - **iv)** the pilot's actions on the aircraft control surfaces.
- **d)** Is this a continuous, sampled-data, or digital control system?
- **e)** Show that it is possible for the pilot of flight 1081 to fly a zigzag course which would show up as a straight line on the controller's screen. What is the (lowest) frequency of a sinusoidal zigzag course which will be hidden from the controller's radar?

1.2

- **a)** Design a continuous lead compensation for the satellite attitude control example described by (1.1) so that the complex roots are at $s = -4.4 \pm j4.4$ rad/sec.
- **b)** Assuming the compensation is to be implemented digitally, approximate the lag from the ZOH to be

$$G_h(s) = \frac{2/T}{s + 2/T},$$

and determine the root locations for sample rates, $\omega_s = 5$ Hz, 10 Hz, and 20 Hz where $T = 1/\omega_s$ sec if ω_s is in units of Hz.

c) The closed-loop system with roots as specified by (a) has a rise time of about 0.3 sec. How fast do you think one should sample in order to have a reasonably smooth response?

1.3 Convert a continuous compensation,

$$D(s) = \frac{u(s)}{e(s)} = \frac{s+a}{s+b},$$

to discrete by writing $D(s)$ as a differential equation and then using the definition of a derivative,

$$\dot{x} = \lim_{T \to 0} \frac{x(k) - x(k-1)}{T},$$

to convert the differential equation to an equation that determines $u(k)$ given $u(k-1)$, $e(k)$, and $e(k-1)$. Assume that T is small enough so that the equation for $\dot{x}$ is approximately correct.

1.4 If a signal varies between 0 and 10 volts (called the *dynamic range*) and it is required that the signal must be represented in the digital computer to the nearest 5 millivolts, that is, if the *resolution* must be 5 mv, determine how many bits the analog-to-digital convertor must have.

CHAPTER 2

Linear, Discrete, Dynamic-Systems Analysis: The z-Transform

2.1 INTRODUCTION

The unique element in the structure of Fig. 1.1 is the digital computer. The fundamental character of the digital computer is that it takes a finite time to compute answers, and it does so with only finite precision. The purpose of this chapter is to develop tools of analysis necessary to understand and to guide the design of programs for a computer acting as a linear, dynamic control component. Needless to say, digital computers can do many things other than control linear, dynamic systems; it is our purpose in this chapter to examine their characteristics when doing this elementary control task and to develop the basic analysis tools needed to write programs for a real-time control computer.

2.2 LINEAR DIFFERENCE EQUATIONS

We assume that the analog-to-digital converter (A/D) in Fig. 1.1 takes samples of the signal y at discrete times and passes them to the computer so that $\hat{y}(kT) = y(kT)$. The job of the computer is to take these sample values and compute in some fashion the signals to be put out through the digital-to-analog converter (D/A). The characteristics of the A/D and D/A converters will be discussed later. Here we consider the treatment of the data inside the computer. Suppose we call the input signals up to the kth sample $e_0, e_1, e_2, \ldots, e_k$, and the output signals prior to that time $u_0, u_1, u_2, \ldots, u_{k-1}$. Then, to get the next output, we have the machine com-

pute some function, which we can express in symbolic form as

$$u_k = f(e_0, \ldots, e_k; u_0, \ldots, u_{k-1}). \tag{2.1}$$

Because we plan to emphasize the elementary and the dynamic possibilities, we assume that the function f in (2.1) is *linear* and depends on only a *finite* number of past e's and u's. Thus we write

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \cdots - a_n u_{k-n} + b_0 e_k + b_1 e_{k-1} + \cdots + b_m e_{k-m}. \tag{2.2}$$

Equation (2.2) is called a linear recurrence equation or difference equation and, as we shall see, has many similarities with a linear differential equation. The name "difference equation" derives from the fact that we could write (2.2) using u_k plus the differences in u_k, which are defined as

$$\begin{aligned}
\nabla u_k &= u_k - u_{k-1} && \text{(first difference)},\\
\nabla^2 u_k &= \nabla u_k - \nabla u_{k-1} && \text{(second difference)},\\
\nabla^n u_k &= \nabla^{n-1} u_k - \nabla^{n-1} u_{k-1} && (n\text{th difference}).
\end{aligned} \tag{2.3}$$

If we solve (2.3) for the values of u_k, u_{k-1}, and u_{k-2} in terms of differences, we find

$$\begin{aligned}
u_k &= u_k,\\
u_{k-1} &= u_k - \nabla u_k,\\
u_{k-2} &= u_k - 2\nabla u_k + \nabla^2 u_k.
\end{aligned}$$

Thus, for a second-order equation with coefficients a_1, a_2, and b_0 (we let $b_1 = b_2 = 0$ for simplicity), we find the equivalent difference equation to be

$$a_2 \nabla^2 u_k - (a_1 + 2a_2)\nabla u_k + (a_2 + a_1 + 1)u_k = b_0 e_k.$$

Although the two forms are equivalent, the recurrence form of (2.2) is more convenient for computer implementation; we will drop the form using differences. We will continue, however, to refer to our equations as "difference equations." If the a's and b's in (2.2) are constant, then the computer is solving a *constant-coefficient difference equation* (CCDE). We plan to demonstrate later that with such equations the computer can control linear constant dynamic systems and approximate most of the other tasks of linear, constant, dynamic systems, including performing the functions of electronic filters. To do so, it is necessary first to examine methods of obtaining solutions to (2.2) and to study the general properties of these solutions.

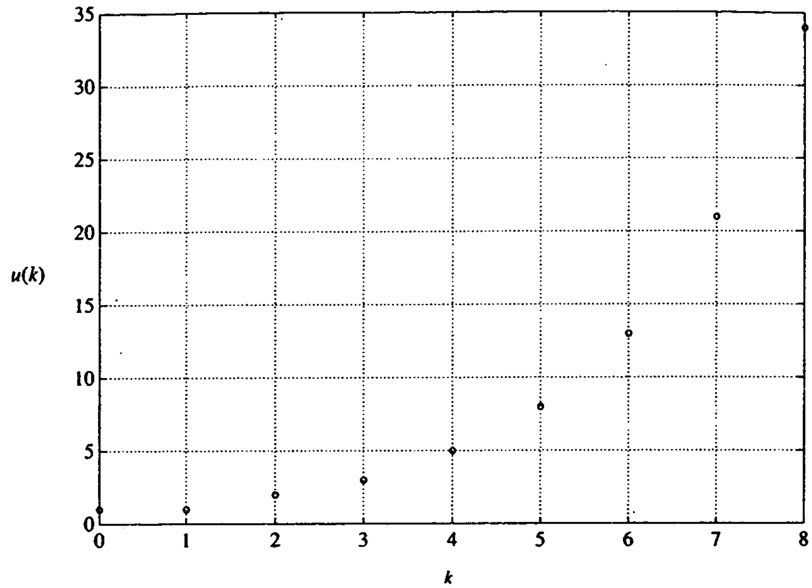


Figure 2.1 The Fibonacci numbers.

To solve a specific CCDE is an elementary matter. We need a starting time (k-value) and some initial conditions to characterize the contents of the computer memory at this time. For example, suppose we take the case

$$u_k = u_{k-1} + u_{k-2} \tag{2.4}$$

and start at $k = 2$. Here there are no input values, and to compute u_2 we need to know the (initial) values for u_0 and u_1. Let us take them to be $u_0 = u_1 = 1$. The first nine values are $1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots$. A plot of the values of u_k versus k is shown in Fig. 2.1.

The results, the Fibonacci numbers, are named after the thirteenth-century mathematician[1] who studied them. For example, (2.4) has been used to model the growth of rabbits in a protected environment.[2] However that may be, the output of the system represented by (2.4) would seem to be

[1]Leonardo Fibonacci of Pisa, who introduced Arabic notation to the Latin world about 1200 A.D.

[2]Wilde (1964). Assume that u_k represents pairs of rabbits and that babies are born in pairs. Assume that no rabbits die and that a new pair begin reproduction after one period. Thus at time k, we have all the old rabbits, u_{k-1}, plus the newborn pairs born to the mature rabbits, which are u_{k-2}.

growing, to say the least. If the response of a dynamic system to any finite initial conditions can grow without bound, we call the system *unstable*. We would like to be able to examine equations like (2.2) and, without having to solve them explicitly, see if they are stable or unstable and even understand the general shape of the solution.

One approach to solving this problem is to assume a form for the solution with unknown constants and to solve for the constants to match the given initial conditions. For continuous, ordinary, differential equations that are constant and linear, exponential solutions of the form e^{st} are used. In the case of linear, constant, difference equations, it turns out that solutions of the form z^k will do where z has the role of s and k is the discrete independent variable replacing time, t. Consider (2.4). If we assume that $u(k) = Az^k$, we get the equation

$$Az^k = Az^{k-1} + Az^{k-2}.$$

Now if we assume $z \neq 0$ and $A \neq 0$, we can divide by A and multiply by z^{-k}, with the result

$$1 = z^{-1} + z^{-2}$$

or

$$z^2 = z + 1.$$

This polynomial of second degree has two solutions, $z = 1/2 \pm \sqrt{5}/2$. Let's call these z_1 and z_2. Since our equation is linear, a sum of the individual solutions will also be a solution. Thus, we have found that a solution to (2.4) is of the form

$$u(k) = A_1 z_1{}^k + A_2 z_2{}^k.$$

We can solve for the unknown constants by requiring that this general solution satisfy the specific initial conditions given. If we substitute $k = 0$ and $k = 1$, we obtain the simultaneous equations

$$1 = A_1 + A_2,$$
$$1 = A_1 z_1 + A_2 z_2.$$

These equations are easily solved[3] to give

$$A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}},$$

$$A_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

And now we have the complete solution of (2.4) in a closed form. Furthermore, we can see that since $z_1 = (1+\sqrt{5})/2$ is greater than 1, the term in ${z_1}^k$ will grow without bound as k grows, which confirms our suspicion that the equation represents an unstable system. We can generalize this result. The equation in z that we obtain after we substitute $u = z^k$ is a polynomial in z known as the *characteristic equation* of the difference equation. If any solution of this equation is outside the unit circle (has a magnitude greater than one), the corresponding difference equation is unstable in the specific sense that for some finite initial conditions the solution will grow without bound as time goes to infinity. If *all* the roots of the characteristic equation are *inside* the unit circle, the corresponding difference equation is stable.

Example 2.1: Is the equation

$$u(k) = 0.9u(k-1) - 0.2u(k-2)$$

stable? The characteristic equation is

$$z^2 - 0.9z + 0.2 = 0,$$

and the characteristic roots are $z = 0.5$ and $z = 0.4$. Since both these roots are inside the unit circle, the equation is stable.

As an example of the origins of a difference equation with an external input, we consider the discrete approximation to integration. Suppose we have a continuous signal, $e(t)$, of which a segment is sketched in Fig. 2.2, and we wish to compute an approximation to the integral

$$\mathcal{J} = \int_0^t e(t)\,dt, \tag{2.5}$$

[3] Do it!

Figure 2.2 Plot of a function and alternative approximations to the area under the curve over a single time interval.

using only the discrete values $e(0), \ldots, e(t_{k-1}), e(t_k)$. We assume that we have an approximation for the integral from zero to the time t_{k-1} and we call it u_{k-1}. The problem is to obtain u_k from this information. Taking the view of the integral as the area under the curve $e(t)$, we see that this problem reduces to finding an approximation to the area under the curve between t_{k-1} and t_k. Three alternatives are sketched in Fig. 2.2. We can use the rectangle of height e_{k-1}, or the rectangle of height e_k, or the trapezoid formed by connecting e_{k-1} to e_k by a straight line. If we take the third choice, the area of the trapezoid is

$$A = \frac{t_k - t_{k-1}}{2}(e_k + e_{k-1}). \tag{2.6}$$

Finally, if we assume that the sampling period, $t_k - t_{k-1}$, is a constant, T, we are led to a simple formula for discrete (trapezoid rule) integration:

$$u_k = u_{k-1} + \frac{T}{2}(e_k + e_{k-1}). \tag{2.7}$$

If $e(t) = t$, then $e_k = kT$ and substitution of $u_k = (T^2/2)k^2$ satisfies (2.7) and is exactly the integral of e. [It should be, because if $e(t)$ is a straight line, the trapezoid is the *exact* area.] If we approximate the area under the curve by the rectangle of height e_{k-1}, the result is called the Forward Rectangular Rule and is described by

$$u_k = u_{k-1} + Te_{k-1}.$$

A third possibility is the Backward Rectangular Rule, given by

$$u_k = u_{k-1} + Te_k.$$

Each of these integration rules is a special case of our general difference equation (2.2). We will examine the properties of these rules later, in Chapter 4, while discussing means to obtain a difference equation that will be equivalent to a given differential equation.

Thus we see that difference equations can be evaluated directly by a digital computer and that they can represent models of physical processes and approximations to integration. It turns out that if the difference equations are linear with coefficients that are constant, we can describe the relation between u and e by a transfer function, and thereby gain a great aid to analysis and also to the design of linear, constant, discrete controls.

2.3 THE DISCRETE TRANSFER FUNCTION

We will obtain the transfer function of linear, constant, discrete systems by the method of z-transform analysis. A logical alternative viewpoint that requires a bit more mathematics but has some appeal is given in Section 2.7.2. The results are the same. We also show how these same results can be expressed in the state space form in Section 2.3.3.

2.3.1 The z-Transform

If a signal has discrete values $e_0, e_1, \ldots, e_k, \ldots$ we define the z-transform of the signal as the function[4,5]

$$\begin{aligned} E(z) &\triangleq \mathcal{Z}\{e_k\} \\ &\triangleq \sum_{k=-\infty}^{\infty} e_k z^{-k}, \qquad r_0 < |z| < R_0, \end{aligned} \tag{2.8}$$

[4] We use the notation $\triangleq$ to mean "is defined as."

[5] In (2.8) the lower limit is $-\infty$ so that values of e_k on both sides of $k = 0$ are included. The transform so defined is sometimes called the two-sided z-transform to distinguish it from the one-sided definition, which would be $\sum_0^\infty e_k z^{-k}$. For signals that are zero for $k < 0$, the transforms obviously give identical results. To take the one-sided transform of u_{k-1}, however, we must handle the value of u_{-1}, and thus are initial conditions introduced by the one-sided transform. Examination of this property and other features of the one-sided transform are invited by the problems. We select the two-sided transform because we need to consider signals that extend into negative time when we study random signals in Chapter 8.

and we assume we can find values of r_0 and R_0 as bounds on the magnitude of the complex variable z for which the series (2.8) converges. A discussion of convergence is deferred until Section 2.7.

Example 2.2: As an example to illustrate (2.8), consider that the data e_k are taken as samples from the time signal $e^{-at}1(t)$ at sampling period T where $1(t)$ is the unit step function, zero for negative t, and one for positive t. Then $e_k = e^{-akT}1(kT)$. The z-transform of this is

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} e_k z^{-k} &= \sum_{0}^{\infty} e^{-akT} z^{-k} \\
&= \sum_{0}^{\infty} (e^{-aT} z^{-1})^k \\
&= \frac{1}{1 - e^{-aT} z^{-1}} \qquad e^{-aT} < |z| < \infty \\
&= \frac{z}{z - e^{-aT}} \qquad |z| > e^{-aT}.
\end{aligned}
$$

We will return to the analysis of signals and development of a table of useful z-transforms in Section 2.5; we first examine the use of the transform to reduce difference equations to algebraic equations and techniques for representing these as block diagrams.

2.3.2 The Transfer Function

The z-transform has the same role in discrete systems that the Laplace transform has in analysis of continuous systems. For example, the z-transforms for e_k and u_k in the difference equation (2.2) or in the trapezoid integration (2.7) are related in a simple way that permits the rapid solution of linear, constant, difference equations of this kind. To find the relation, we proceed by direct substitution. We take the definition given by (2.8) and, in the same way, we define the z-transform of the sequence $\{u_k\}$ as

$$
U(z) \triangleq \sum_{k=-\infty}^{\infty} u_k z^{-k}. \tag{2.9}
$$

Now we multiply (2.7) by z^{-k} and sum over k. We get

$$\sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} u_{k-1} z^{-k} + \frac{T}{2}\left(\sum_{k=-\infty}^{\infty} e_k z^{-k} + \sum_{k=-\infty}^{\infty} e_{k-1} z^{-k}\right). \tag{2.10}$$

From (2.9), we recognize the left-hand side as $U(z)$. In the first term on the right, we let $k-1=j$ to obtain

$$\sum_{k=-\infty}^{\infty} u_{k-1} z^{-k} = \sum_{j=-\infty}^{\infty} u_j z^{-(j+1)} = z^{-1}U(z). \tag{2.11}$$

By similar operations on the third and fourth terms we can reduce (2.10) to

$$U(z) = z^{-1}U(z) + \frac{T}{2}[E(z) + z^{-1}E(z)]. \tag{2.12}$$

Equation (2.12) is now simply an algebraic equation in z and the functions U and E. Solving it we obtain

$$U(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}E(z). \tag{2.13}$$

We *define* the ratio of the transform of the output to the transform of the input as the *transfer function*, $H(z)$. Thus, in this case, the transfer function for trapezoid-rule integration is

$$\frac{U(z)}{E(z)} \triangleq H(z) = \frac{T}{2}\frac{z+1}{z-1}. \tag{2.14}$$

For the more general relation given by (2.2), it is readily verified by the same techniques that

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}},$$

and if $n \geq m$, we can write this as a ratio of polynomials in z as

$$\begin{aligned} H(z) &= \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} \\ &= \frac{b(z)}{a(z)}. \end{aligned} \tag{2.15}$$

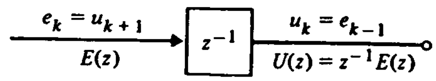


Figure 2.3 The unit delay.

The general input–output relation between transforms with linear, constant, difference equations is

$$U(z) = H(z)E(z). \tag{2.16}$$

Although we have developed the transfer function with the z-transform, it is also true that the transfer function is the ratio of the output to the input when both vary as z^k.

Because $H(z)$ is a rational function of a complex variable, we use the terminology of that subject. Suppose we call the numerator polynomial $b(z)$ and the denominator $a(z)$. The places in z where $b(z) = 0$ are *zeros* of the transfer function, and the places in z where $a(z) = 0$ are the *poles* of $H(z)$. If z_0 is a pole and $(z - z_0)^p H(z)$ has neither pole nor zero at z_0, we say that $H(z)$ has a pole of order p at z_0. If $p = 1$, the pole is simple. The transfer function (2.14) has a simple pole at $z = 1$ and a simple zero at $z = -1$.

We can now give a physical meaning to the variable z. Suppose we let all coefficients in (2.15) be zero except b_1 and we take b_1 to be 1. Then $H(z) = z^{-1}$. But $H(z)$ represents the transform of (2.2), and with these coefficient values the difference equation reduces to

$$u_k = e_{k-1}. \tag{2.17}$$

The present value of the output, u_k, equals the input *delayed by one period.* Thus we see that a transfer function of z^{-1} is a *delay* of one time unit. We can picture the situation as in Fig. 2.3, where both time and transform relations are shown.

Since the relations of (2.7), (2.14), and (2.15) are all composed of delays, they can be expressed in terms of z^{-1}! Consider (2.7). In Fig. 2.4 we illustrate the difference equation (2.7) using the transfer function z^{-1} as the symbol for a unit delay.

We can follow the operations of the discrete integrator by tracing the signals through Fig. 2.4. For example, the present value of e_k is passed to the first summer, where it is added to the previous value e_{k-1}, and the sum is multiplied by $T/2$ to compute the area of the trapezoid between e_{k-1} and e_k. This is the signal marked a_k in Fig. 2.4. After this, there is another sum, where the previous output, u_{k-1}, is added to the new area to form the next

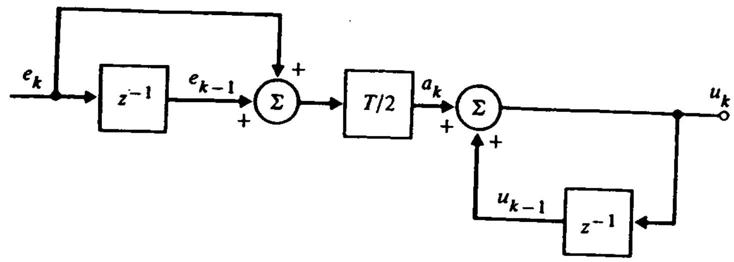


Figure 2.4 A block diagram of trapezoid integration as represented by (2.7).

value of the integral estimate, u_k. The discrete integration occurs in the loop with one delay, z^{-1}, and unity gain.

2.3.3 Block Diagrams and State-Variable Descriptions

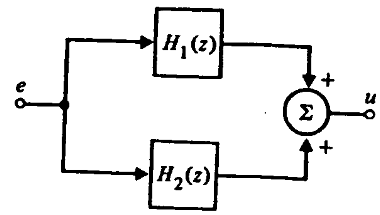
Because (2.16) is a linear algebraic relationship, a system of such relations is described by a system of linear equations. These can be solved by the methods of linear algebra or by the graphical methods of block diagrams. To use block-diagram analysis to manipulate these discrete-transfer-function relationships, there are only four primitive cases:

1. The transfer function of paths in parallel is the sum of the single-path transfer functions (Fig. 2.5).
2. The transfer function of paths in series is the *product* of the path transfer functions (Fig. 2.6).
3. The transfer function of a single loop of paths is the transfer function of the forward path divided by one minus the loop transfer function (Fig. 2.7).
4. The transfer function of an arbitrary multipath diagram is given by combinations of these cases. Mason's rule[6] can also be used.

For the general difference equation of (2.2), we already have the transfer function in (2.15). It is interesting to connect this case with a block diagram *using only simple delay forms for* z in order to see several "canonical" block diagrams and to introduce the description of discrete systems using equations of state.

There are many ways to reduce the difference equation (2.2) to a block diagram involving z only as the delay operator, z^{-1}. The first one we will

[6]Mason (1956). See Franklin, Powell, and Emami-Naeini(1986) for a discussion.



$H(z) = H_1(z) + H_2(z)$

Figure 2.5 Block diagram of parallel blocks.

consider leads to the "control" canonical form. We begin with the transfer function as a ratio of polynomials

$$U(z) = H(z)E(z) = \frac{b(z)}{a(z)}E(z) = b(z)\xi,$$

where

$$\xi = \frac{E(z)}{a(z)}$$

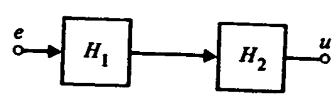
and thus

$$a(z)\xi = E(z).$$

At this point we need to get specific; and rather than carry through with a system of arbitrary order, we will work out the details for the third-order case and leave it to the reader to extend the results in the obvious way to whatever order is desired. In the development that follows, we will consider the variables u, e, and ξ as *time* variables and z as an advance operator such that $zu(k) = u(k+1)$ or $z^{-1}u = u(k-1)$. With this convention (which is simply using the property of z derived earlier), consider the equations

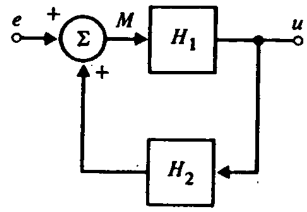
$$(z^3 + a_1z^2 + a_2z + a_3)\xi = e, \tag{2.18}$$

$$(b_0z^3 + b_1z^2 + b_2z + b_3)\xi = u. \tag{2.19}$$



$H(z) = H_1(z)H_2(z)$

Figure 2.6 Block diagram of cascade blocks.



$$H(z) = \frac{H_1(z)}{1 - H_1(z)H_2(z)}$$

Figure 2.7 Feedback transfer function.

We can write (2.18) as

$$\begin{aligned} z^3\xi &= e - a_1 z^2 \xi - a_2 z \xi - a_3 \xi \\ \xi(k+3) &= e(k) - a_1 \xi(k+2) - a_2 \xi(k+1) - a_3 \xi(k). \end{aligned} \tag{2.20}$$

Now assume we have $z^3\xi$, which is to say that we have $\xi(k+3)$ because z^3 is an advance operator of three steps. If we operate on this with z^{-1} three times in a row, we will get back to $\xi(k)$, as shown in Fig. 2.8(a). From (2.20), we can now compute $z^3\xi$ from e and the lower powers of z and ξ given in the block diagram; the picture is now as given in Fig. 2.8(b). To complete the representation of (2.18) and (2.19), we need only add the formation of the output u as a weighted sum of the variables $z^3\xi$, $z^2\xi$, $z\xi$, and ξ according to (2.19). The completed picture is shown in Fig. 2.8(c).

In Fig 2.8(c), the internal variables have been named x_1, x_2, and x_3. These variables comprise the *state* of this dynamic system in this form. Having the block diagram shown in Fig. 2.8(c), we can write down, almost by inspection, the difference equations that describe the evolution of the state, again using the fact that the transfer function z^{-1} corresponds to a one-unit delay. For example, we see that $x_3(k+1) = x_2(k)$ and $x_2(k+1) = x_1(k)$. Finally, expressing the sum at the far left of the figure, we have

$$x_1(k+1) = -a_1 x_1(k) - a_2 x_2(k) - a_3 x_3(k) + e(k).$$

We collect these three equations together in proper order, and we have

$$\begin{aligned} x_1(k+1) &= -a_1 x_1(k) - a_2 x_2(k) - a_3 x_3(k) + e(k), \\ x_2(k+1) &= x_1(k), \\ x_3(k+1) &= x_2(k). \end{aligned} \tag{2.21}$$

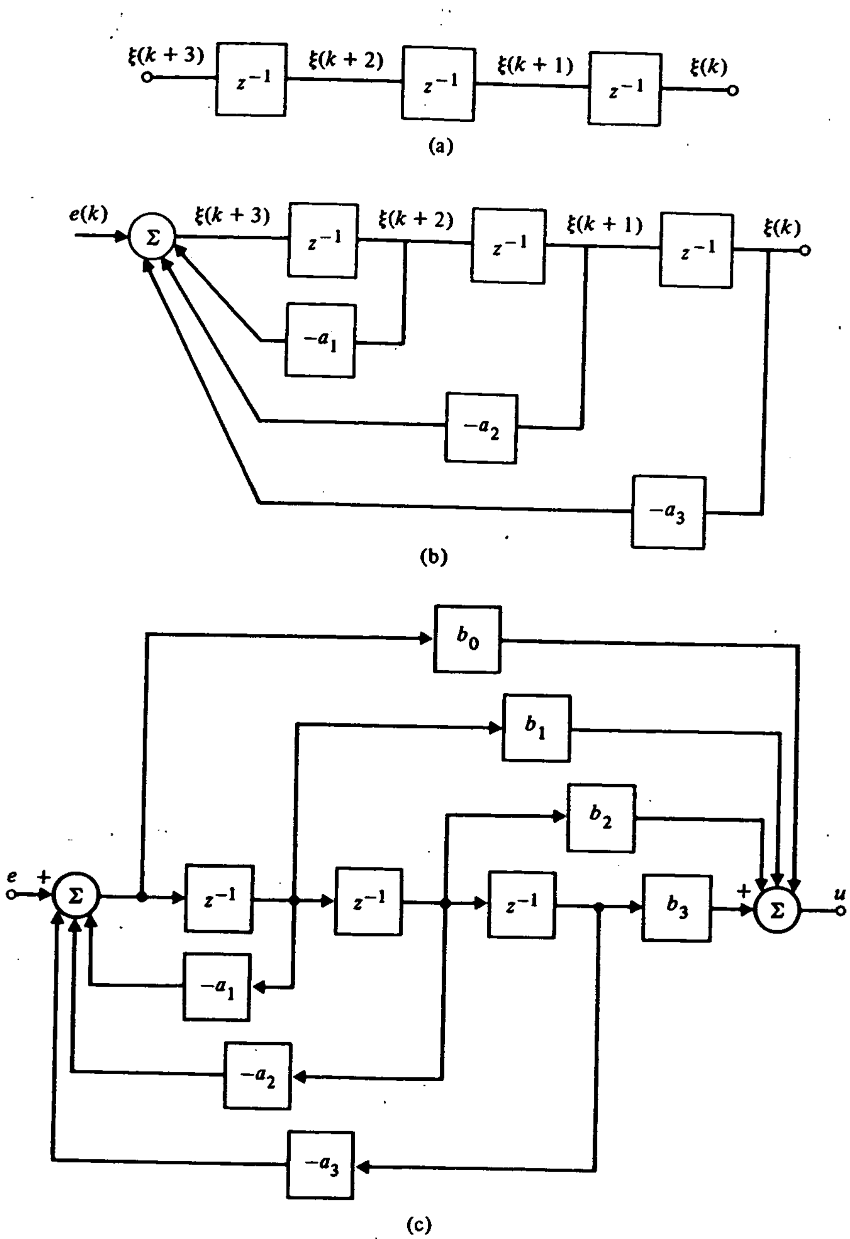


Figure 2.8 Block diagram development of control canonical form. (a) Solving for $\xi(k)$; (b) solving for $\xi(k+3)$ from $e(k)$ and past ξ's; (c) solving for $U(k)$ from ξ's.

Using vector-matrix notation,[7] we can write this in the compact form

$$\mathbf{x}(k+1) = \mathbf{A}_c\mathbf{x}(k) + \mathbf{B}_c e(k),$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{2.22a}$$

and

$$\mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.22b}$$

The output equation is also immediate except that we must watch to catch *all* paths by which the state variables combine in the output. The problem is caused by the b_0 term. If $b_0 = 0$, then $u = b_1x_1 + b_2x_2 + b_3x_3$, and the corresponding matrix form is immediate. However, if b_0 is not 0, x_1 for example not only reaches the output through b_1 but also by the parallel path with gain $-b_0a_1$. The complete equation is

$$u = (b_1 - a_1b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_3 - a_3b_0)x_3) + b_0e.$$

In vector/matrix notation, we have

$$u = \mathbf{C}_c x + \mathbf{D}_c e,$$

where

$$\mathbf{C}_c = [\, b_1 - a_1b_0 \quad b_2 - a_2b_0 \quad b_3 - a_3b_0 \,], \tag{2.23a}$$

$$\mathbf{D}_c = [\, b_0 \,]. \tag{2.23b}$$

[7] We assume the reader has some knowledge of matrices. The results we require and references to study material are given in Appendix C. To distinguish vectors and matrices from scalar variables, we will use bold-face type.

We can combine the equations for the state evolution and the output to give the very useful and most compact equations for the dynamic system,

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}_c\mathbf{x}(k) + \mathbf{B}_c e(k), \\ u(k) &= \mathbf{C}_c x(k) + \mathbf{D}_c e(k), \end{aligned} \tag{2.24}$$

where $\mathbf{A}_c$ and $\mathbf{B}_c$ for this control canonical form are given by (2.22), and $\mathbf{C}_c$ and $\mathbf{D}_c$ are given by (2.23).

The other canonical form we want to illustrate is called the observer canonical form and is found by starting with the difference equations in operator/transform form as

$$z^3u + a_1z^2u + a_2zu + a_3u = b_0z^3e + b_1z^2e + b_2ze + b_3e.$$

In this equation, the external input is $e(k)$, and the response is $u(k)$, which is the solution of this equation. The terms with factors of z are time-shifted toward the future with respect to k and must be eliminated in some way. To do this, we assume at the start that we have the $u(k)$, and of course the $e(k)$, and we rewrite the equation as

$$b_3e - a_3u = z^3u + a_1z^2u + a_2zu - b_0z^3e - b_1z^2e - b_2ze.$$

Here, *every* term on the right is multiplied by at least one power of z, and thus we can operate on the lot by z^{-1} as shown in the partial block diagram drawn in Fig. 2.9(a).

Now in this internal result there appear a_2u and $-b_2e$, which can be cancelled by adding proper multiples of u and e, as shown in Fig. 2.9(b), and once they have been removed, the remainder can again be operated on by z^{-1}.

If we continue this process of subtracting out the terms at k and operating on the rest by z^{-1}, we finally arrive at the place where all that is left is u alone! But that is just what we assumed we had in the first place, so connecting this term back to the start finishes the block diagram, which is drawn in Fig. 2.9(c).

A preferred choice of numbering for the state components is also shown in the figure. Following the technique used for the control form, we find that the matrix equations are given by

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}_O\mathbf{x}(k) + \mathbf{B}_O e(k), \\ u(k) &= \mathbf{C}_O x(k) + \mathbf{D}_O e(k), \end{aligned} \tag{2.25}$$

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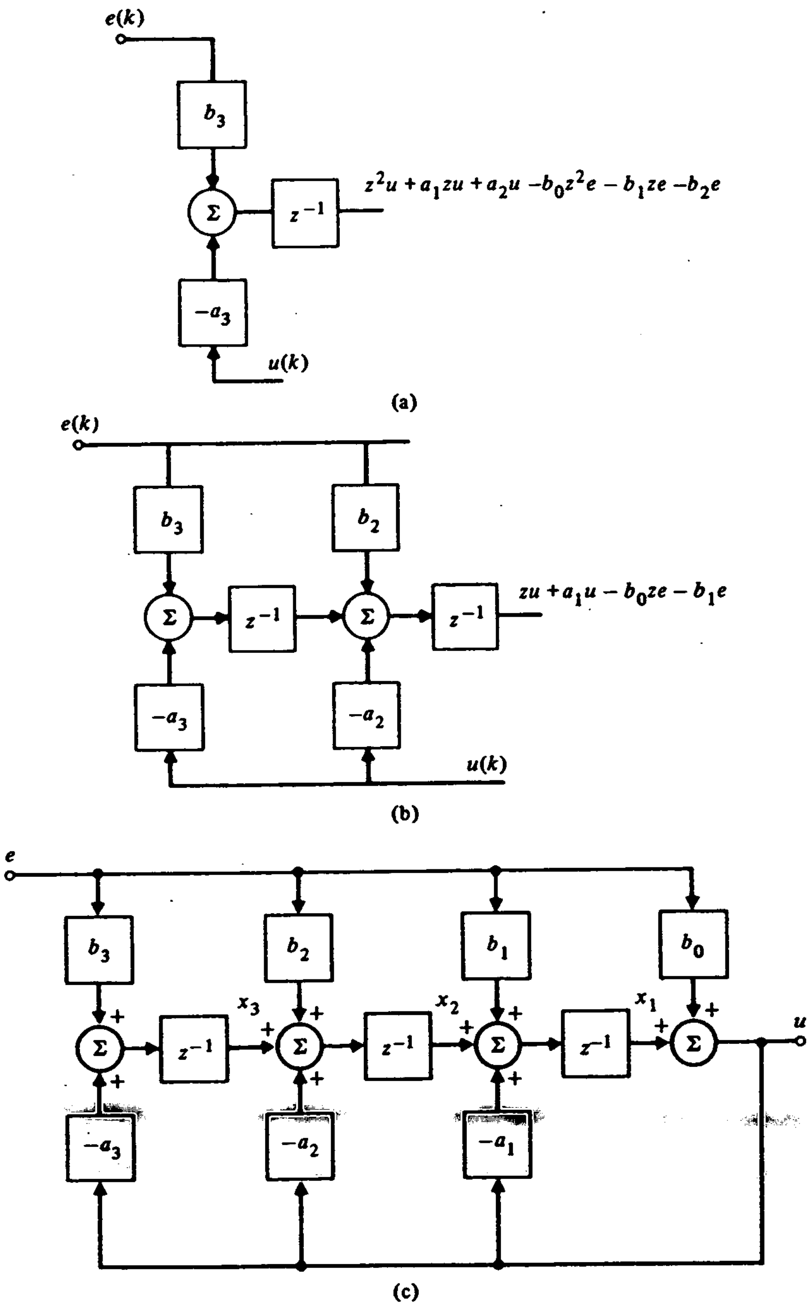


Figure 2.9 Block diagram development of observer canonical form. (a) The first partial sum and delay; (b) the second partial sum and delay; (c) the completion with the solution for $u(k)$.

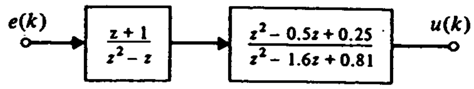


Figure 2.10 Block diagram of a cascade realization.

where

$$\mathbf{A}_0 = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}_0 = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ b_3 - b_0 a_3 \end{bmatrix},$$

$$\mathbf{C}_0 = [1 \quad 0 \quad 0],$$

$$\mathbf{D}_0 = [b_0].$$

The block diagrams of Figs. 2.8 and 2.9 are called *direct canonical* realizations of the transfer function $H(z)$ because the gains of the realizations are coefficients in the transfer-function polynomials. Another useful form is obtained if we realize a transfer function by placing several first- or second-order direct forms in series with each other, a *cascade canonical* form. In this case, the $H(z)$ is represented as a product of factors, and the poles and zeros of the transfer function are clearly represented in the coefficients.

For example, suppose we have a transfer function

$$\begin{aligned} H(z) &= \frac{z^3 + 0.5z^2 - 0.25z + 0.25}{z^4 - 2.6z^3 + 2.4z^2 - 0.8z} \\ &= \frac{(z+1)(z^2 - 0.5z + 0.25)}{(z^2 - z)(z^2 - 1.6z + 0.8)}. \end{aligned}$$

The zero factor $z+1$ can be associated with the pole factor $z^2 - z$ to form one second-order system, and the zero factor $z^2 - 0.5z + 0.25$ can be associated with the second-order pole factor $z^2 - 1.6z + 0.8$ to form another. The cascade factors, which could be realized in a direct form such as control or observer form, make a cascade form as shown in Fig. 2.10.

2.3.4 Relation of Transfer Function to Pulse Response

We have shown that a transfer function of z^{-1} is a unit delay in the time domain. We can also give a time-domain meaning to an arbitrary transfer func-

Table 2.1 Step-by-step construction of the unit pulse response for Fig. 2.4.

k	e_{k-1}	e_k	a_k	u_{k-1}	$u_k \equiv h_k$
0	0	1	$T/2$	0	$T/2$
1	1	0	$T/2$	$T/2$	T
2	0	0	0	T	T
3	0	0	0	T	T

tion. Recall that the z-transform is defined by (2.8) to be $E(z) = \Sigma e_k z^{-k}$, and the transfer function is defined from (2.16) as $H(z)$ when the input and output are related by $U(z) = H(z)E(z)$. Now suppose we deliberately select $e(k)$ to be the unit discrete pulse defined by

$$\begin{aligned} e_k &= \begin{cases} 1, & (k = 0), \\ 0, & (k \neq 0), \end{cases} \\ &\triangleq \delta_k. \end{aligned} \tag{2.26}$$

Then it follows that $E(z) = 1$ and therefore that

$$U(z) = H(z). \tag{2.27}$$

Thus the transfer function $H(z)$ is seen to be the *transform* of the response to a unit-pulse input. For example, let us look at the system of Fig. 2.4 and put a unit pulse in at the e_k-node (with no signals in the system beforehand).[8] We can readily follow the pulse through the block and build Table 2.1.

Thus the unit-pulse response is zero for negative k, is $T/2$ at $k = 0$, and equals T thereafter. The z-transform of this sequence is

$$H(z) = \sum_{-\infty}^{\infty} u_k z^{-k} \triangleq \sum_{-\infty}^{\infty} h_k z^{-k}.$$

[8] In this development we assume that (2.7) is intended to be used as a formula for computing values of u_k as k *increases*. There is no reason why we could not also solve for u_k as k takes on negative values. The direction of time comes from the application and not from the recurrence equation.

$$
\begin{array}{llll}
e_0 \; +e_1z^{-1} & +e_2z^{-2} & +e_32^{-3} & +\cdots \\
h_0 \; +h_1z^{-1} & +h_2z^{-3} & +h_32^{-3} & +\cdots \\
\hline
e_0h_0+e_1h_0z^{-1} & +e_2h_0z^{-2} & +e_3h_0z^{-3} & \\
\quad +e_0h_1z^{-1} & +e_1h_1z^{-2} & +e_2h_1z^{-3} & \\
 & +e_0h_2z^{-2} & +e_1h_2z^{-3} & \\
 & & +e_0h_3z^{-3} & \\
\hline
\multicolumn{4}{l}{e_0h_0+(e_0h_1+e_1h_0)z^{-1}+(e_0h_2+e_1h_1+e_2h_0)z^{-2}+(e_0h_3+e_1h_2+e_2h_1+e_3h_0)z^{-3}+\cdots}
\end{array}
$$

Figure 2.11 Representation of the product $E(z)H(z)$ as a product of polynomials.

If we add $T/2$ to the z^0-term and subtract $T/2$ from the whole series, we have a simpler sum, as follows:

$$
\begin{aligned}
H(z) &= \sum_{k=0}^{\infty} Tz^{-k} - \frac{T}{2} \\
&= \frac{T}{1-z^{-1}} - \frac{T}{2} \qquad (1 < |z|) \\
&= \frac{2T - T(1-z^{-1})}{2(1-z^{-1})} \\
&= \frac{T + Tz^{-1}}{2(1-z^{-1})} \\
&= \frac{T}{2}\frac{z+1}{z-1} \qquad (1 < |z|).
\end{aligned} \tag{2.28}
$$

Of course, this is the transfer function we obtained in (2.13) from direct analysis of the difference equation.

A final point of view useful in the interpretation of the discrete transfer function is obtained by multiplying the infinite polynomials of $E(z)$ and $H(z)$ as suggested in (2.16). For purposes of illustration, we will assume that the unit-pulse response, h_k, is zero for $k < 0$. Likewise, we will take $k = 0$ to be the starting time for e_k. Then the product that produces $U(z)$ is the polynomial product given in Fig. 2.11.

Since this product has been shown to be $U(z) = \Sigma u_k z^{-k}$, it must therefore follow that the coefficient of z^{-k} in the product is u_k. Listing these

coefficients, we have the relations

$$\begin{aligned}
u_0 &= e_0h_0,\\
u_1 &= e_0h_1 + e_1h_0,\\
u_2 &= e_0h_2 + e_1h_1 + e_2h_0,\\
u_3 &= e_0h_3 + e_1h_2 + e_2h_1 + e_3h_0.
\end{aligned}$$

The extrapolation of this simple pattern gives the result

$$u_k = \sum_{j=0}^{k} e_j h_{k-j}.$$

By extension, we let the lower limit of the sum be $-\infty$ and the upper limit be $+\infty$:

$$u_k = \sum_{j=-\infty}^{\infty} e_j h_{k-j}. \tag{2.29}$$

Negative values of j in the sum correspond to inputs applied before time equals zero. Values for j greater than k occur if the unit-pulse response is nonzero for negative arguments. By definition, such a system, which responds *before* the input that causes it occurs, is called *noncausal.* This is the discrete convolution sum and is the analog of the convolution integral that relates input and impulse response to output in linear, constant, continuous systems.

To verify (2.29) we can take the z-transform of both sides:

$$\sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} z^{-k} \sum_{j=-\infty}^{\infty} e_j h_{k-j}.$$

Interchanging the sum on j with the sum on k leads to

$$U(z) = \sum_{j=-\infty}^{\infty} e_j \sum_{k=-\infty}^{\infty} z^{-k} h_{k-j}.$$

Now let $k - j = l$ in the second sum:

$$U(z) = \sum_{j=-\infty}^{\infty} e_j \sum_{l=-\infty}^{\infty} h_l z^{-(l+j)},$$

but $z^{-(l+j)} = z^{-l}z^{-j}$, which leads to

$$U(z) = \sum_{j=-\infty}^{\infty} e_j z^{-j} \sum_{l=-\infty}^{\infty} h_l z^{-l},$$

and we recognize these two separate sums as

$$U(z) = E(z)H(z).$$

We can also derive the convolution sum from the properties of linearity and stationarity. First we need more formal definitions of "linear" and "stationary."

1. *Linearity:* A system with input e and output u is *linear* if superposition applies, which is to say, if $u_1(k)$ is the response to $e_1(k)$ and $u_2(k)$ is the response to $e_2(k)$, then the system is linear if and only if, for every scalar α and β, the response to $\alpha e_1 + \beta e_2$ is $\alpha u_1 + \beta u_2$.
2. *Stationarity:* A system is *stationary*, or time invariant, if a time shift in the input results in only a time shift in the output. For example, if we take the system at rest (no internal energy in the system) and apply a certain signal $e(k)$, suppose we observe a response $u(k)$. If we repeat this experiment at any later time when the system is again at rest and we apply the shifted input, $e(k-N)$, if we see $u(k-N)$, then the system is stationary.

These properties can be used to derive the convolution in (2.29) as follows. If response to a unit pulse at $k = 0$ is $h(k)$, then response to a pulse of intensity e_0 is $e_0 h(k)$ if the system is linear. Furthermore, if the system is *constant,* then a delay of the input will delay the response. Thus, if

$$e = \begin{cases} e_l, & k = l, \\ 0, & k \neq l, \end{cases}$$

then the response will be $e_l h_{k-l}$.

Finally, by linearity again, the total response at time k to a sequence of these pulses is the *sum* of the responses, namely,

$$u_k = e_0 h_k + e_1 h_{k-1} + \cdots + e_l h_{k-l} + \cdots + e_k h_0,$$

or

$$u_k = \sum_{l=0}^{k} e_l h_{k-l}.$$

Now note that if the input sequence began in the distant past, we must include terms for $l < 0$, perhaps back to $l = -\infty$. Similarly, if the system should be noncausal, future values of e where $l > k$ may also come in. The general case is thus (again)

$$u_k = \sum_{l=-\infty}^{\infty} e_l h_{k-l}. \tag{2.30}$$

2.3.5 External Stability and Jury's Test

A very important qualitative property of a dynamic system is stability, and we can consider internal or external stability. Internal stability is concerned with the responses at all the internal variables such as those that appear at the delay elements in a canonical block diagram as in Fig. 2.8 or Fig. 2.9 (the state). Otherwise we can be satisfied to consider only the *external stability* as given by the study of the input–output relation described for the linear stationary case by the convolution (2.30). These differ in that some internal modes might not be connected to both the input and the output of a given system.

For external stability, the most common definition of *appropriate response* is that for every Bounded Input, we should have a Bounded Output. If this is true we say the system is BIBO stable. A test for BIBO stability can be given directly in terms of the unit-pulse response, h_k. First we consider a sufficient condition. Suppose the input e_k is bounded, that is, there is an M such that

$$|e_l| \le M < \infty \qquad \text{for all } l. \tag{2.31}$$

If we consider the magnitude of the response given by (2.30), it is easy to see that

$$|u_k| \le \left|\sum e_l h_{k-l}\right|,$$

which is surely less than the sum of the magnitudes as given by

$$\le \sum_{-\infty}^{\infty} |e_l|\,|h_{k-l}|.$$

But, because we assume (2.31), this result is in turn bounded by

$$\le M \sum_{-\infty}^{\infty} |h_{k-l}|. \tag{2.32}$$

Thus the output will be bounded for every bounded input if

$$\sum_{l=-\infty}^{\infty} |h_{k-l}| < \infty. \tag{2.33}$$

This condition is also necessary, for if we consider the bounded (by 1!) input

$$\begin{aligned} e_l &= \frac{h_{-l}}{|h_{-l}|} \qquad (h_{-l} \neq 0) \\ &= 0 \qquad\qquad (h_{-l} = 0) \end{aligned}$$

and apply it to (2.30), the output at $k = 0$ is

$$\begin{aligned} u_0 &= \sum_{l=-\infty}^{\infty} e_l h_{-l} \\ &= \sum_{l=-\infty}^{\infty} \frac{(h_{-l})^2}{|h_{-l}|} \\ &= \sum_{l=-\infty}^{\infty} |h_{-l}|. \end{aligned} \tag{2.34}$$

Thus, unless the condition given by (2.34) is true, the system is not BIBO stable.

Example 2.3: The test given by (2.34) can be applied to the unit pulse response used to compute (2.13) and given as the u_k-column in Table 2.1 on page 31:

$$\begin{aligned} h_0 &= T/2, \\ h_k &= T, \qquad k > 0, \end{aligned}$$

$$\sum |h_k| = T/2 + \sum_1^{\infty} T = \text{unbounded}. \tag{2.35}$$

Thus this discrete approximation to integration is not (BIBO) stable!

Example 2.4: As a second example, we consider the difference equation (2.2) with all coefficients except a_1 and b_0 equal to zero:

$$u_k = a_1 u_{k-1} + b_0 e_k. \tag{2.36}$$

The unit-pulse response is easily developed from the first few terms to be

$$\begin{aligned} &u_0 = b_0, \qquad u_1 = a_1 b_0, \qquad u_2 = a_1^2 b_0, \dots \\ &u_k = h_k = b_0 a^k, \qquad k \geq 0. \end{aligned} \tag{2.37}$$

Applying the test, we have

$$\begin{aligned} \sum_{-\infty}^{\infty} |h_l| = \sum_{\infty=0}^{\infty} b_0 |a^l| &= b_0 \frac{1}{1-|a|} \qquad (|a| < 1) \\ &= \text{unbounded} \qquad (|a| \geq 1). \end{aligned}$$

Thus we conclude that the system described by this equation is BIBO stable if $|a| < 1$, and unstable otherwise.

For a more general rational transfer function with many simple poles, we can expand the function in partial fractions about its poles, and the corresponding pulse response will be a sum of respective terms. As we saw earlier, if a pole is inside the unit circle, the corresponding pulse response decays with time geometrically and is stable. Thus, if all poles are inside the unit circle, the system with rational transfer function is stable; if at least one pole is on or outside the unit circle, the corresponding system is not BIBO stable. With modern computer programs available, finding the poles of a particular transfer function is no big deal. Sometimes, however, we wish to test for stability of an entire class of systems; or, as in an adaptive control system, the potential poles are constantly changing and we wish to have a quick test for stability in terms of the literal polynomial coefficients. In the continuous case, such a test was provided by Routh; in the discrete case, the most convenient such test was worked out by Jury and Blanchard(1961).

The Jury test is in the same spirit as the Routh test (see Franklin, Powell, and Emami-Naeini for a discussion) in that we form two rows from the coefficients of length n, and from these, by a series of two-by-two determinants, we compute a successor row of length $n-1$. With this reduced

length row we form another successor of length $n-2$ and so on until we have a row of length 1. The test consists of examining the sign of the first entries in selected rows. As with the Routh test, the Jury test is much more difficult to derive than to use; here we illustrate only the use.

If we have a transfer function $H(z) = b(z)/a(z)$, then this system will be stable if and only if all roots of $a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ are inside the unit circle. To test for this condition by the Jury test, multiply $a(z)$ by -1 if necessary to make the sign of a_0 positive. Then form rows of the coefficients, the even rows being in reversed order, as follows:

$$\begin{array}{cccc} a_0 & a_1 & \cdots & a_n \\ a_n & a_{n-1} & \cdots & a_0 \\ b_0 & b_1 & \cdots & . \\ b_{n-1} & b_{n-2} & \cdots & . \end{array}$$

The entries in the third row are formed from the second-order determinants using the first column of the first two rows with each of the other columns from these rows *starting from the right* and dividing by a_0. The result can be expressed by the formulas:

$$b_0 = a_0 - \frac{a_n}{a_0} a_n,$$

$$b_1 = a_1 - \frac{a_n}{a_0} a_{n-1},$$

$$b_k = a_k - \frac{a_n}{a_0} a_{n-k},$$

The elements in the third row are reversed to form the fourth row and the process is repeated. For example, the elements of the fifth row are given by

$$c_k = b_k - \frac{b_{n-1}}{b_0} b_{n-1-k}.$$

The original polynomial is stable (has all roots inside the unit circle) if all the terms in the first columns of the odd rows are positive, that is, if $a_0 > 0, b_0 > 0, c_0 > 0, \ldots$.

This test is readily implemented in a computer program.[9]

[9]See STABLE in Table E.1 in Appendix E.

Example 2.5: To illustrate the use of Jury's test, we consider first the simple second-order polynomial

$$a(z) = z^2 + a_1 z + a_2.$$

The Jury array is

$$\begin{array}{ccc}
1 & a_1 & a_2 \\
a_2 & a_1 & 1 \\
1 - a_2^2 & a_1 - a_1 a_2 & . \\
a_1 - a_1 a_2 & 1 - a_2^2 & . \\
\dfrac{(1 - a_2^2)^2 - a_1^2(1 - a_2)^2}{1 - a_2^2} & . & .
\end{array}$$

From row three, we have the condition that $1 - {a_2}^2 > 0$, and from this we conclude that

$$-1 < a_2 < 1.$$

From row five, we can factor out $(1 - a_2)^2$ to conclude that

$$(1 + a_2)^2 > {a_1}^2$$

and thus,

$$a_2 + 1 > a_1 \qquad \text{and} \qquad a_2 + 1 > -a_1.$$

From these inequalities, we can draw the *stability triangle* shown in Fig 2.12.

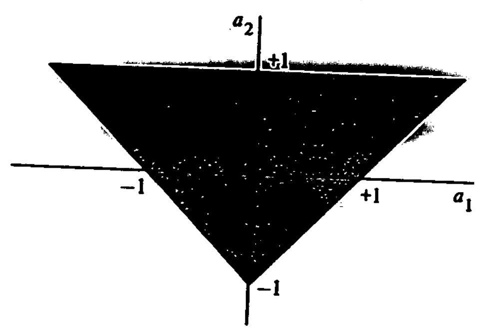


Figure 2.12 Stability triangle for the general, real, second-order polynomial.

Example 2.6: For the next example, consider the polynomial

$$a(z) = z^3 - 2.1z^2 + 1.6z - 0.4.$$

The Jury array is

$$\begin{array}{cccc}
1 & -2.1 & 1.6 & -0.4 \\
-0.4 & 1.6 & -2.1 & 1 \\
0.84 & -1.46 & 0.76 & . \\
0.76 & -1.46 & 0.84 & . \\
0.1524 & -0.139 & . & . \\
-0.139 & 0.1524 & . & . \\
0.0256 & . & . & .
\end{array}$$

The test is from the odd rows: $1 > 0, 0.84 > 0, 0.1524 > 0$, $0.0256 > 0$, and we conclude that a system with this polynomial as its denominator would be stable. As a matter of fact, the poles are at $z = 0.5$ and $z = 0.8 \pm 0.4j$.

Example 2.7: As a third example, consider the polynomial

$$z^3 - 2.6z^2 + 2.4z - 0.8$$

The odd rows only of the Jury array are

$$\begin{array}{cccc}
1 & -2.6 & 2.4 & -0.8 \\
0.36 & -0.68 & 0.32 & . \\
0.0756 & -0.0756 & . & . \\
0 & . & . & .
\end{array}$$

From these computations we conclude that the polynomial does *not* have all its roots inside the unit circle; and, because the last term is zero and a small perturbation would send it either way, inside or outside, there must be at least one root exactly on the unit circle. In fact, the roots are $z = 1$ and $z = 0.8 \pm 0.4j$.

As an aid to testing stability, it can be shown that it is *necessary* for a stable polynomial (with positive first term) that the polynomial evaluated

at $z = 1$ and $z = -1$ must both be positive. The first value is just the sum of the coefficients, and the second is the sum with alternating sign changes. These two tests are quickly done and can save time if the only purpose is to be sure that the system is stable; many unstable systems will be rejected by these simple tests without going through the entire Jury array. In the previous case, for instance, the sum of coefficients is zero and we need go no further; the polynomial cannot be stable.

2.4 DISCRETE MODELS OF SAMPLED-DATA SYSTEMS

The systems and signals we have studied thus far have been defined in discrete time only. Most of the dynamic systems to be controlled, however, are continuous systems and, if linear, are described by continuous transfer functions in the Laplace variable s. The interface between the continuous and discrete domains are the A/D and the D/A converters as shown in Fig. 1.1. In this section we develop the analysis needed to compute the discrete transfer function between the samples that come from the digital computer to the D/A converter and the samples that are picked up by the A/D converter.[10] The situation is drawn in Fig. 2.13.

2.4.1 Using the z-Transform

We wish to find the discrete transfer function from the input samples $u(kT)$ (which probably come from a computer of some kind) to the output samples, $y(kT)$ picked up by the A/D converter. Although it is possibly confusing at first, we follow convention and call the discrete transfer function $G(z)$ when the continuous transfer function is $G(s)$. Although $G(z)$ and $G(s)$ are entirely different functions, they do describe the *same* plant, and the use of s for the continuous transform and z for the discrete transform is always maintained. To find $G(z)$ we need only observe that the $y(kT)$ are samples of the plant output when the input is from the D/A converter. As for the D/A converter, we assume that this device, commonly called a zero-order hold or ZOH, accepts a sample $u(kT)$ at $t = kT$ and holds its output constant

[10] In Chapter 3, a comprehensive frequency analysis of sampled data systems is presented. Here we undertake only the special problem of finding the sample-to-sample discrete transfer function of a continuous system between a D/A and an A/D.

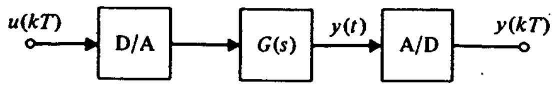


Figure 2.13 The prototype sampled-data system.

at this value until the next sample is sent at $t = kT + T$. The piecewise constant output of the D/A is the signal, $u(t)$, that is applied to the plant.

Our problem is now really quite simple because we have just seen that the discrete transfer function is the z-transform of the samples of the output when the input samples are the unit pulse at $k = 0$. If $u(kT) = 1$ for $k = 0$ and $u(kT) = 0$ for $k \neq 0$, the output of the D/A converter is a pulse of width T seconds and height 1, as sketched in Fig. 2.14. Mathematically, this pulse is given by $1(t) - 1(t - T)$. Let us call the particular output in response to the pulse shown in Fig. 2.14 $y_1(t)$. This response is the difference between the step response [to $1(t)$] and the delayed step response [to $1(t - T)$]. The Laplace transform of the step response is $G(s)/s$. Thus in the transform domain the unit pulse response of the plant is

$$Y_1(s) = (1 - e^{-Ts})\frac{G(s)}{s}, \tag{2.38}$$

and the required transfer function is the z-transform of the samples of the inverse of $Y_1(s)$, which can be expressed as

$$\begin{aligned} G(z) &= \mathcal{Z}\{y_1(kT)\} \\ &= \mathcal{Z}\{\mathcal{L}^{-1}\{Y_1(s)\}\} \triangleq \mathcal{Z}\{Y_1(s)\} \\ &= \mathcal{Z}\left\{(1 - e^{-Ts})\frac{G(s)}{s}\right\}. \end{aligned}$$

This is the sum of two parts. The first is $\mathcal{Z}\{G(s)/s\}$, and the second is

$$\mathcal{Z}\{e^{-Ts}G(s)/s\} = z^{-1}\mathcal{Z}\{G(s)/s\}$$

because e^{-Ts} is exactly a delay of one period. Thus the transfer function is

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \tag{2.39}$$

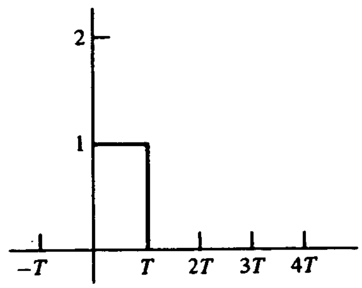


Figure 2.14 D/A output for unit-pulse input.

Example 2.8: As a first example of computing such discrete transfer functions, suppose $G(s) = a/(s+a)$. Then

$$\frac{G(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a},$$

and the corresponding time function is

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = 1(t) - e^{-at}1(t).$$

The samples of this signal are $1(kT) - e^{-akT}1(kT)$, and the z-transform of these samples is

$$\begin{aligned} \mathcal{Z}\left\{\frac{G(s)}{s}\right\} &= \frac{z}{z-1} - \frac{z}{z-e^{-aT}} \\ &= \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}. \end{aligned}$$

We could have gone to the tables in Appendix B and found this result directly as entry 12. Now we can compute the desired transform by applying (2.39)

$$\begin{aligned} G(z) &= \frac{z-1}{z}\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} \\ &= \frac{1-e^{-aT}}{z-e^{-aT}}. \end{aligned} \tag{2.40}$$

Example 2.9: We consider the double integrator characteristic of a single mass, such as the satellite, for which the transfer function is $G(s) = 1/s^2$. We have

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s^3}\right\}$$

This time we refer to the tables in Appendix B and find that the transform of $1/s^3$ is

$$\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3},$$

and therefore

$$G(z) = \frac{T^2(z+1)}{2(z-1)^2}. \qquad (2.41)$$

For more complex systems than those in Examples 2.8 and 2.9, use of a CAD package is recommended.[11]

2.4.2 Continuous Time Delay

We now consider computing the discrete transfer function of a continuous system with pure time delay. The responses of many chemical process-control plants exhibit pure time delay because there is a finite time of transport of fluids or materials between the process and the controls and/or the sensors. Also, we must often consider finite computation time in the digital controller, and this is exactly the same as if the process had a pure time delay. With the techniques we have developed here, it is possible to obtain the discrete transfer function of such processes exactly, as Example 2.10 illustrates.

Example 2.10: We consider the example suggested by the fluid mixer problem described in Appendix A.3, for which

$$G(s) = e^{-\lambda s}H(s).$$

[11]See X-C2D in Table E.1.

The term $e^{-\lambda s}$ represents the delay of λ seconds, which includes both the process delay and the computation delay, if any. We assume that $H(s)$ is a rational transfer function. To prepare this function for computation of the z-transform, we first define an integer ℓ and a positive number m less than 1.0 such that $\lambda = \ell T - mT$. With these definitions we can write

$$\frac{G(s)}{s} = e^{-\ell T s} \frac{e^{mTs} H(s)}{s}.$$

Because ℓ is an integer, this term reduces to $z^{-\ell}$ when we take the z-transform. Because $m < 1$, the transform of the other term is quite direct. We select $H(s) = a/(s+a)$ and, after the partial fraction expansion of $H(s)$, we have

$$G(z) = \frac{z-1}{z^{\ell+1}} \mathcal{Z} \left\{ \frac{e^{mTs}}{s} - \frac{e^{mTs}}{s+a} \right\}$$

To complete the transfer function, we need the z-transforms of the inverses of the terms in the braces. The first term is a unit step shifted left by mT seconds, and the second term is an exponential shifted left by the same amount. Because $m < 1$, these shifts are less than one full period, and no sample is picked up in negative time. The signals are sketched in Fig. 2.15.

The samples are given by $1(kT)$ and $e^{-aT(k+m)}1(kT)$. The corresponding z-transforms are $z/(z-1)$ and $ze^{-amT}/(z-e^{-aT})$. Consequently the final transfer function is

$$\begin{aligned} G(z) &= \frac{z-1}{z}\frac{1}{z^{\ell}} \left\{ \frac{z}{z-1} - \frac{ze^{-amT}}{z-e^{-aT}} \right\} \\ &= \frac{z-1}{z} \left\{ \frac{z[z-e^{-aT}-(z-1)e^{-amT}]}{(z-1)(z-e^{-aT})} \right\} \\ &= (1-e^{-amT}) \frac{z+\alpha}{z^{\ell}(z-e^{-aT})}, \end{aligned}$$

where the zero position is at $-\alpha = -(e^{-amT} - e^{-aT})/(1-e^{-amT})$. Notice that this zero is near the origin of the z-plane when m is near 1 and moves outside the unit circle to near $-\infty$ when m approaches 0. For the specific values of the mixer, we take $a = 1$, $T = 1$, and

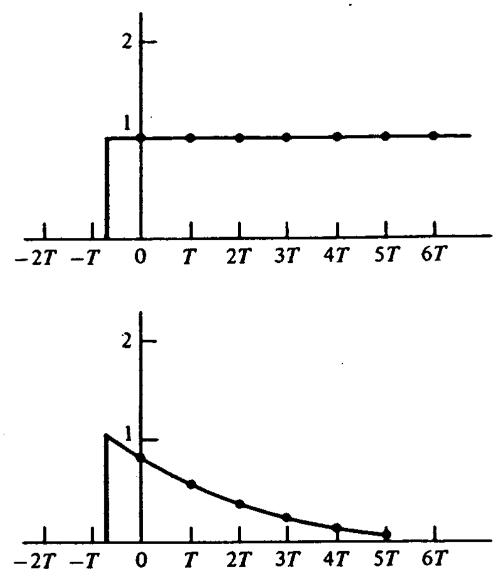


Figure 2.15 Sketch of the shifted signals showing sample points.

$\lambda = 1.5$. Then we can compute that $\ell = 2$ and $m = 0.5$. For these values, we get

$$G(z) = \frac{z + 0.6065}{z^2(z - 0.3679)}. \tag{2.42}$$

2.4.3 State-Space Form

Computing the z-transform using the Laplace transform as in (2.39) is a very tedious business that is unnecessary with the availability of computers. We will next develop a formula using state descriptions that will remove most of the calculations to the computer, where it is better done. A continuous, linear, constant-coefficient system of differential equations can always be expressed as a set of first-order matrix differential equations:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u + \mathbf{G}_1 w, \tag{2.43}$$

where u is the control input to the system and w is a disturbance input. The output can be expressed as a linear combination of the state, $\mathbf{x}$, and the input as

$$y = \mathbf{H}\mathbf{x} + Ju. \tag{2.44}$$

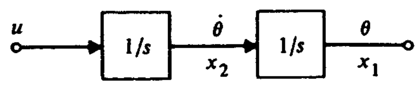


Figure 2.16 Satellite attitude control in classical representation.

Often the sampled-data system being described is the plant of a control problem, and the parameter J in (2.44) is zero and will frequently be omitted.

Example 2.11: Application of state representation to the equations of the satellite attitude-control example shown in Fig. 2.16 and described in Appendix A yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{G}} u,$$
$$\theta = y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.45}$$

which, in this case, turns out to be a rather involved way of writing

$$\ddot{\theta} = u.$$

The representations (2.43) and (2.44) are not unique. Given one state representation, any nonsingular linear transformation of that state such as $\boldsymbol{\xi} = \mathbf{Tx}$ is also an allowable alternative realization of the same system.

If we let $\boldsymbol{\xi} = \mathbf{Tx}$ in (2.43) and (2.44), we find

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{T}\dot{\mathbf{x}} = \mathbf{T}(\mathbf{Fx} + \mathbf{G}u + \mathbf{G}_1 w) \\ &= \mathbf{TFx} + \mathbf{TG}u + \mathbf{TG}_1 w, \\ \dot{\boldsymbol{\xi}} &= \mathbf{TFT}^{-1}\boldsymbol{\xi} + \mathbf{TG}u + \mathbf{TG}_1 w, \\ y &= \mathbf{HT}^{-1}\boldsymbol{\xi} + Ju. \end{aligned}$$

If we designate the system matrices for the new state $\boldsymbol{\xi}$ as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, then

$$\dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \mathbf{B}u + \mathbf{B}_1 w, \qquad y = \mathbf{C}\boldsymbol{\xi} + \mathbf{D}u,$$

Figure 2.17 System definition with sampling operations shown.

where

$$\mathbf{A} = \mathbf{T}\mathbf{F}\mathbf{T}^{-1}, \quad \mathbf{B} = \mathbf{T}\mathbf{G}, \quad \mathbf{B}_1 = \mathbf{T}\mathbf{G}_1, \quad \mathbf{C} = \mathbf{H}\mathbf{T}^{-1}, \quad D = J.$$

Example 2.12: As an illustration, we can let $\xi_1 = x_2$ and $\xi_2 = x_1$ in (2.45); or, in matrix notation, the transformation to interchange the states is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case $\mathbf{T}^{-1} = \mathbf{T}$, and application of the transformation equations to the system matrices of (2.45) gives

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = [0 \quad 1].$$

Most often, a change of state is made to bring the description matrices into a useful canonical form. We saw earlier how a single high-order difference equation could be represented by a state description in control or in observer canonical form. Also, there is a very useful state description corresponding to the partial-fraction expansion of a transfer function. State transformations can take a general description for either a continuous or a discrete system and, subject to some technical restrictions, convert it into a description in one or the other of these forms, as needed.

We wish to use the state description to establish a general method for obtaining the difference equations that represent the behavior of the continuous plant. Fig. 2.17 again depicts the portion of our system under consideration. Ultimately, the digital controller will take the samples $y(k)$, operate on that sequence by means of a difference equation, and put out a sequence of numbers, $u(k)$, which are the inputs to the plant. The loop will, therefore, be closed. To analyze the result, we must be able to relate the samples of the output $y(k)$ to the samples of the control $u(k)$. To do this, we must solve (2.43).

We will solve the general equation in two steps. We begin by solving the equation with only initial conditions and no external input. This is the homogeneous equation

$$\dot{\mathbf{x}}_h = \mathbf{F}\mathbf{x}_h(t), \qquad \mathbf{x}_h(t_0) = \mathbf{x}_0 \tag{2.46}$$

To solve this, we assume the solution is sufficiently smooth that a series expansion of the solution is possible:

$$\mathbf{x}_h(t) = A_0 + A_1(t - t_0) + A_2(t - t_0)^2 + \cdots. \tag{2.47}$$

If we let $t = t_0$, we find immediately that $A_0 = \mathbf{x}_0$. If we differentiate (2.47) and substitute into (2.46), we have

$$A_1 + 2A_2(t - t_0) + 3A_3(t - t_0)^2 + \cdots = \mathbf{F}\mathbf{x}_h$$

and, at $t = t_0$, $A_1 = \mathbf{F}\mathbf{x}_0$. Now we continue to differentiate the series and the differential equation and equate them at t_0 to arrive at the series

$$\mathbf{x}_h(t) = \left[\mathbf{I} + \mathbf{F}(t - t_0) + \frac{\mathbf{F}^2(t - t_0)^2}{2} + \frac{\mathbf{F}^3(t - t_0)^3}{6} + \cdots\right]\mathbf{x}_0.$$

This series is defined as the matrix exponential and written

$$\mathbf{x}_h(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0), \tag{2.48}$$

where, by definition, the matrix exponential is

$$\begin{aligned} e^{\mathbf{F}(t-t_0)} &= \mathbf{I} + \mathbf{F}(t - t_0) + \mathbf{F}^2\frac{(t - t_0)^2}{2!} + \mathbf{F}^3\frac{(t - t_0)^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \mathbf{F}^k \frac{(t - t_0)^k}{k!}. \end{aligned} \tag{2.49}$$

It can be shown that the solution given by (2.48) is unique, which leads to very interesting properties of the matrix exponential. For example, consider two values of t : t_1 and t_2. We have

$$\mathbf{x}(t_1) = e^{\mathbf{F}(t_1-t_0)}\mathbf{x}(t_0)$$

and

$$\mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_0)}\mathbf{x}(t_0).$$

Because t_0 is arbitrary also, we can express $\mathbf{x}(t_2)$ as if the equation solution began at t_1, for which

$$\mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_1)}x(t_1).$$

Substituting for $\mathbf{x}(t_1)$ gives

$$\mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_1)}e^{\mathbf{F}(t_1-t_0)}\mathbf{x}(t_0).$$

We now have two separate expressions for $\mathbf{x}(t_2)$, and, if the solution is unique, these must be the same. Hence we conclude that

$$e^{\mathbf{F}(t_2-t_0)} = e^{\mathbf{F}(t_2-t_1)}e^{\mathbf{F}(t_1-t_0)} \tag{2.50}$$

for all t_2, t_1, t_0. Note especially that if $t_2 = t_0$, then

$$\mathbf{I} = e^{-\mathbf{F}(t_1-t_0)}e^{\mathbf{F}(t_1-t_0)}.$$

Thus we can obtain the inverse of $e^{\mathbf{F}t}$ by merely changing the sign of t! We will use this result in computing the particular solution to (2.43).

The particular solution when u is not zero is obtained by using the method of *variation of parameters*.[12] We guess the solution to be in the form

$$\mathbf{x}_p(t) = e^{\mathbf{F}(t-t_0)}\mathbf{v}(t), \tag{2.51}$$

where $\mathbf{v}(t)$ is a vector of variable parameters to be determined [as contrasted to the constant parameters $\mathbf{x}(t_0)$ in (2.48)]. Substituting (2.51) into (2.43), we obtain

$$\mathbf{F}e^{\mathbf{F}(t-t_0)}\mathbf{v} + e^{\mathbf{F}(t-t_0)}\dot{\mathbf{v}} = \mathbf{F}e^{\mathbf{F}(t-t_0)}\mathbf{v} + \mathbf{G}u,$$

and, using the fact that the inverse is found by changing the sign of the exponent, we can solve for $\dot{\mathbf{v}}$ as

$$\dot{\mathbf{v}}(t) = e^{-\mathbf{F}(t-t_0)}\mathbf{G}u(t).$$

---

[12] Due to Joseph Louis Lagrange, French mathematician (1736–1813). We assume $w = 0$, but because the equations are linear, the effect of w can be added later.

Assuming that the control $u(t)$ is zero for $t < t_0$, we can integrate $\dot{\mathbf{v}}$ from t_0 to t to obtain

$$\mathbf{v}(t) = \int_{t_0}^{t} e^{-\mathbf{F}(\tau - t_0)} \mathbf{G} u(\tau) d\tau.$$

Hence, from (2.51), we get

$$\mathbf{x}_p(t) = e^{\mathbf{F}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{F}(\tau - t_0)} \mathbf{G} u(\tau) d\tau,$$

and simplifying, using the results of (2.50), we obtain the particular solution (convolution)

$$\mathbf{x}_p(t) = \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)} \mathbf{G} u(\tau) d\tau. \tag{2.52}$$

The total solution for $w = 0$ and $u \neq 0$ is the sum of (2.48) and (2.52):

$$\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)} \mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)} \mathbf{G} u(\tau) d\tau. \tag{2.53}$$

We wish to use this solution over one sample period to obtain a difference equation: hence we juggle the notation a bit (let $t = kT + T$ and t_0 equal kT) and arrive at a particular version of (2.53):

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T} \mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)} \mathbf{G} u(\tau) d\tau. \tag{2.54}$$

This result is not dependent on the type of hold because u is specified in terms of its continuous time history, $u(t)$, over the sample interval. A common and typically valid assumption is that of a zero-order hold (ZOH) with no delay, that is,

$$u(\tau) = u(kT), \qquad kT \leq \tau < kT + T.$$

If some other hold is implemented or if there is a delay between the application of the control from the ZOH and the sample point, this fact can be accounted for in the evaluation of the integral in (2.54). The equations for a delayed ZOH will be given in the next subsection. To facilitate the solution

of (2.54) for a ZOH with no delay, we change variables in the integral from τ to η such that

$$\eta = kT + T - \tau.$$

Then we have

$$\mathbf{x}(kT + T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G} u(kT). \tag{2.55}$$

If we define

$$\mathbf{\Phi} = e^{\mathbf{F}T}, \tag{2.56a}$$

$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G}, \tag{2.56b}$$

Eqs. (2.55) and (2.44) reduce to difference equations in standard form:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma} u(k) + \mathbf{\Gamma}_1 w(k), \\ y(k) &= \mathbf{H}\mathbf{x}(k), \end{aligned} \tag{2.57}$$

where we include the effect of an impulsive or piecewise constant disturbance, w, and assume that $J = 0$ in this case. If w is a constant, then $\mathbf{\Gamma}_1$ is given by (2.56b) with $\mathbf{G}$ replaced by $\mathbf{G}_1$. If w is an impulse, then $\mathbf{\Gamma}_1 = \mathbf{G}_1$.[13] The $\mathbf{\Phi}$ series expansion

$$\mathbf{\Phi} = e^{\mathbf{F}T} = \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2 T^2}{2!} + \frac{\mathbf{F}^3 T^3}{3!} + \cdots,$$

can also be written

$$\mathbf{\Phi} = \mathbf{I} + \mathbf{F}T\mathbf{\Psi}, \tag{2.58}$$

where

$$\mathbf{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2 T^2}{3!} + \cdots$$

[13] If $w(t)$ is *not* a function of only its sample values, then an integral like that of (2.54) is required to describe its influence on $\mathbf{x}(k+1)$. Random disturbances are treated in Chapter 9.

1. Select sampling period T and description matrices $\mathbf{F}$ and $\mathbf{G}$.
2. Matrix $\mathbf{I} \leftarrow$ Identity
3. Matrix $\mathbf{\Psi} \leftarrow I$
4. $k \leftarrow 11$ [We are using $N = 11$ in (2.60).]
5. If $k = 1$, go to step 9.
6. Matrix $\mathbf{\Psi} \leftarrow \mathbf{I} + \dfrac{\mathbf{F}T}{k}\mathbf{\Psi}$
7. $k \leftarrow k - 1$
8. Go to step 5.
9. Matrix $\mathbf{\Gamma} \leftarrow T\mathbf{\Psi}\mathbf{g}$
10. Matrix $\mathbf{\Phi} \leftarrow \mathbf{I} + \mathbf{F}T\mathbf{\Psi}$

Figure 2.18 Program logic to compute $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ from $\mathbf{F}$, $\mathbf{G}$, and T for simple cases. (The left arrow, $\leftarrow$, is to be read "is replaced by.")

The $\mathbf{\Gamma}$ integral in (2.56) can be evaluated term by term to give

$$\begin{aligned}\mathbf{\Gamma} &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^{k+1}}{(k+1)!}\mathbf{G} \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^k}{(k+1)!} T\mathbf{G} \\ &= \mathbf{\Psi} T\mathbf{G}. \end{aligned} \tag{2.59}$$

We evaluate $\mathbf{\Psi}$ by a series in the form

$$\mathbf{\Psi} \approx \mathbf{I} + \frac{\mathbf{F}T}{2}\left(\mathbf{I} + \frac{\mathbf{F}T}{3}\left(\cdots \frac{\mathbf{F}T}{N-1}\left(\mathbf{I} + \frac{\mathbf{F}T}{N}\right)\right)\cdots\right), \tag{2.60}$$

which has better numerical properties than the direct series of powers. We then find $\mathbf{\Gamma}$ from (2.59) and $\mathbf{\Phi}$ from (2.58). A discussion of the selection of N and a technique to compute $\mathbf{\Psi}$ for comparatively large T is given by Källström (1973), and a review of various methods is found in a classic paper by Moler and Van Loan (1978). The program logic for computation of $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ for simple cases is given in Fig. 2.18. All control design packages that we know of contain logic to compute $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ from the continuous matrices $\mathbf{F}$, $\mathbf{G}$, and the sample period T.[14]

[14] See X-C2D in Table E.1 in Appendix E.

To compare this method of representing the plant with the discrete transfer functions, we can take the z-transform of (2.57) with $w = 0$ and obtain

$$[z\mathbf{I} - \mathbf{\Phi}]\mathbf{X}(z) = \mathbf{\Gamma} U(z), \tag{2.61a}$$

$$Y(z) = \mathbf{H}\mathbf{X}(z); \tag{2.61b}$$

therefore

$$\frac{Y(z)}{U(z)} = \mathbf{H}[z\mathbf{I} - \mathbf{\Phi}]^{-1}\mathbf{\Gamma}. \tag{2.62}$$

Example 2.13: For the satellite attitude-control example, the $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ matrices are easy to calculate using (2.58) and (2.59) and the values for $\mathbf{F}$ and $\mathbf{G}$ defined in (2.45). Since $\mathbf{F}^2 = 0$ in this case, we have

$$\begin{aligned}
\mathbf{\Phi} &= \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2T^2}{2!} + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \\
\mathbf{\Gamma} &= \left[\mathbf{I}T + \mathbf{F}\frac{T^2}{2!} + \frac{\mathbf{F}^2T^3}{3!} \right] \mathbf{G} \\
&= \left\{ \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{T^2}{2} \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix};
\end{aligned}$$

hence, using (2.61), we obtain

$$\begin{aligned}
\frac{Y(z)}{U(z)} &= [1 \quad 0] \left\{ z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \\
&= \frac{T^2}{2} \frac{(z+1)}{(z-1)^2},
\end{aligned}$$

which is the same result that would be obtained using (2.39) and the z-transform tables.

Note that to compute Y/U we find that the denominator is the determinant $\det(z\mathbf{I} - \mathbf{\Phi})$, which comes from the matrix inverse in (2.62). This

determinant is the characteristic polynomial of the transfer function, and the zeros of the determinant are the poles of the plant. We have two poles at $z = 1$ in this case, corresponding to the two integrations in this plant's equations of motion.

We can explore further the question of poles and zeros and the state-space description by considering again the transform equations (2.61). An interpretation of transfer-function poles from the perspective of the corresponding difference equation is that a pole is a value of z such that the equation has a nontrivial solution when the forcing input is zero. From (2.61a), this implies that the linear eigenvalue equations

$$[z\mathbf{I} - \mathbf{\Phi}]\mathbf{X}(z) = [0]$$

have a nontrivial solution. From matrix algebra the well-known requirement for this is that $\det(z\mathbf{I} - \mathbf{\Phi}) = 0$. In the present case, we have

$$\begin{aligned}
\det[z\mathbf{I} - \mathbf{\Phi}] &= \det\left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\right] \\
&= \det\begin{bmatrix} z-1 & T \\ 0 & z-1 \end{bmatrix} \\
&= (z-1)^2 = 0,
\end{aligned}$$

which is the characteristic equation, as we have seen. To compute the poles numerically when the matrices are given, one would use an eigenvalue routine.[15]

Along the same line of reasoning, a system zero is a value of z such that the system output is zero even with a nonzero state-and-input combination. Thus if we are able to find a nontrivial solution for $\mathbf{X}(z_0)$ and $U(z_0)$ such that $Y(z_0)$ is zero, then z_0 is a zero of the system. Combining the two parts of (2.57), we must satisfy the requirement

$$\begin{bmatrix} z\mathbf{I} - \mathbf{\Phi} & -\mathbf{\Gamma} \\ \mathbf{H} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}(z) \\ U(z) \end{bmatrix} = [0]. \tag{2.63}$$

Once more the condition for the existence of nontrivial solutions is that the determinant of the square coefficient system matrix be zero.[16] For the

[15] See EIGENV in Table E.1.

[16] We do not consider here the case of different numbers of inputs and outputs.

satellite example, we have

$$\det\begin{bmatrix} z-1 & -T & -T^2/2 \\ 0 & z-1 & -T \\ 1 & 0 & 0 \end{bmatrix} = 1\cdot\det\begin{bmatrix} -T & -T^2/2 \\ z-1 & -T \end{bmatrix}$$

$$= +T^2 + \left(\frac{T^2}{2}\right)(z-1)$$

$$= +\frac{T^2}{2}z + \frac{T^2}{2}$$

$$= +\frac{T^2}{2}(z+1).$$

Thus we have a single zero at $z = -1$, as we have seen from the transfer function. Again, to compute the values of the zeros, called transmission zeros, good algorithms exist in matrix algebra.[17]

2.4.4 State-Space Models for Systems with Delay

Thus far we have discussed the calculation of discrete state models from continuous, ordinary differential equations of motion. Now we present the formulas for including a time delay in the model and also a time prediction up to one period which corresponds to the modified z-transform as defined by Jury. We begin with a state-variable model that includes a delay in control action. The state equations are

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}\mathbf{x}(t) + \mathbf{G}u(t-\lambda), \\ y &= \mathbf{H}\mathbf{x}. \end{aligned} \tag{2.64}$$

The general solution to (2.64) is given by (2.54); it is

$$\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau-\lambda)\,d\tau.$$

If we let $t_0 = kT$ and $t = kT + T$, then

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau-\lambda)\,d\tau.$$

[17] See ZEROS in Table E.1. In using this function, one must be careful to account properly for the zeros that are at infinity; the function might return them as very large numbers that the user must remove to uncover the finite zeros.

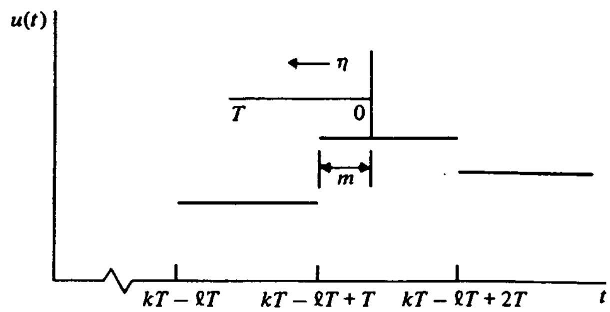


Figure 2.19 Sketch of a piecewise input and time axis for a system with time delay.

If we substitute $\eta = kT + t - \tau$ for τ in the integral, we find a modification of (2.55):

$$\begin{aligned}\mathbf{x}(kT+T) &= e^{\mathbf{F}T}\mathbf{x}(kT) + \int_T^0 e^{\mathbf{F}\eta}\mathbf{G}u(kT+T-\lambda-\eta)(-d\eta)\\ &= e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^T e^{\mathbf{F}\eta}\mathbf{G}u(kT+T-\lambda-\eta)\,d\eta.\end{aligned}$$

If we now separate the system delay λ into an integral number of sampling periods plus a fraction, we can define an integer ℓ and a positive number m less than one such that

$$\lambda = \ell T - mT, \tag{2.65}$$

and

$$\ell \geq 0,$$
$$0 \leq m < 1.$$

With this substitution, we find that the discrete system is described by

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^T e^{\mathbf{F}\eta}\mathbf{G}u(kT+T-\ell T+mT-\eta)\,d\eta. \tag{2.66}$$

If we sketch a segment of the time axis near $t = kT - \ell T$ (Fig. 2.19), the nature of the integral in (2.66) with respect to the variable η will become clear. The integral runs for η from 0 to T, which corresponds to t from $kT - \ell T + T + mT$ backward to $kT - \ell T + mT$. Over this period, the control, which we assume is piecewise constant, takes on first the value $u(kT - \ell T + T)$

and then the value $u(kT-\ell T)$. Therefore, we can break the integral in (2.66) into two parts as follows:

$$\begin{aligned}\mathbf{x}(kT+T) &= e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^{mT} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta\, u(kT-\ell T+T)\\ &\quad + \int_{mT}^{T} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta\, u(kT-\ell T)\\ &= \mathbf{\Phi}\mathbf{x}(kT) + \mathbf{\Gamma}_1 u(kT-\ell T) + \mathbf{\Gamma}_2 u(kT-\ell T+T). \qquad (2.67)\end{aligned}$$

In (2.67) we defined

$$\mathbf{\Phi} = e^{\mathbf{F}T}, \quad \mathbf{\Gamma}_1 = \int_{mT}^{T} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta, \quad \text{and} \quad \mathbf{\Gamma}_2 = \int_0^{mT} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta. \qquad (2.68)$$

To complete our analysis it is necessary to express (2.67) in standard state-space form. To do this we must consider separately the cases of $\ell = 0$, $\ell = 1$, and $\ell > 1$.

For $\ell = 0$, $\lambda = -mT$ according to (2.65), which implies not delay but prediction. Because mT is restricted to be less than T, however, the output will not show a sample before $k = 0$, and the discrete system will be causal. The result is that the discrete system computed with $\ell = 0$, $m \neq 0$ will show the response at $t = 0$, which the same system with $\ell = 0$, $m = 0$ would show at $t = mT$. In other words, by taking $\ell = 0$ and $m \neq 0$ we pick up the response values *between* the normal sampling instants. In z-transform theory, the transform of the system with $\ell = 0$, $m \neq 0$ is called the *modified z-transform.*[18] The state-variable form requires that we evaluate the integrals in (2.68). To do so we first convert $\mathbf{\Gamma}_1$ to a form similar to the integral for $\mathbf{\Gamma}_2$. From (2.68) we factor out the constant matrix G to obtain

$$\mathbf{\Gamma}_1 = \int_{mT}^{T} e^{\mathbf{F}\eta}\, d\eta\, \mathbf{G}.$$

If we set $\sigma = \eta - mT$ in this integral, we have

$$\begin{aligned}\mathbf{\Gamma}_1 &= \int_0^{T-mT} e^{\mathbf{F}(mT+\sigma)} d\sigma \mathbf{G}\\ &= e^{\mathbf{F}m}\int_0^{T-mT} e^{\mathbf{F}\sigma} d\sigma \mathbf{G}. \qquad (2.69)\end{aligned}$$

[18] See Jury (1964) or Ogata (1987).

For notational purposes we will define, for any positive nonzero scalar number, a, the two matrices

$$\mathbf{\Phi}(a) = e^{\mathbf{F}a}, \quad \mathbf{\Psi}(a) = \frac{1}{a}\int_0^a e^{\mathbf{F}\sigma} d\sigma. \tag{2.70}$$

In terms of these matrices, we have

$$\begin{aligned} \mathbf{\Gamma}_1 &= (T - mT)\mathbf{\Phi}(mT)\mathbf{\Psi}(T - mT)\mathbf{G}, \\ \mathbf{\Gamma}_2 &= mT\mathbf{\Psi}(mT)\mathbf{G}. \end{aligned} \tag{2.71}$$

The definition (2.70) is also useful from a computational point of view. If we recall the series definition of the matrix exponential,

$$\mathbf{\Phi}(a) = e^{\mathbf{F}a} = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k a^k}{k!},$$

then we get

$$\begin{aligned} \mathbf{\Psi}(a) &= \frac{1}{a}\int_0^a \sum_{k=0}^{\infty} \frac{\mathbf{F}^k \sigma^k}{k!} d\sigma \\ &= \frac{1}{a}\sum_{k=0}^{\infty} \frac{\mathbf{F}^k}{k!} \frac{a^{k+1}}{k+1} \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k a^k}{(k+1)!}. \end{aligned} \tag{2.72}$$

But now we note that the series for $\mathbf{\Phi}(a)$ can be written as

$$\mathbf{\Phi}(a) = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{F}^k a^k}{k!}.$$

If we let $k = j + 1$ in the sum, then, as in (2.58), we have

$$\begin{aligned} \mathbf{\Phi}(a) &= I + \sum_{j=0}^{\infty} \frac{\mathbf{F}^{j+1} a^{j+1}}{(j+1)!} \\ &= I + \sum_{j=0}^{\infty} \frac{\mathbf{F}^j a^j}{(j+1)!} aF \\ &= I + a\mathbf{\Psi}(a)\mathbf{F}. \end{aligned} \tag{2.73}$$

The point of (2.73) is that only the series for $\Psi(a)$ needs to be computed,The point of (2.73) is that only the series for $\Psi(a)$ needs to be and from this single sum we can compute $\mathbf{\Phi}$ and $\mathbf{\Gamma}$.

If we return to the case $\ell = 0$, $m \neq 0$, the discrete state equations are

$$\mathbf{x}(k+1) = \mathbf{\Phi x}(k) + \mathbf{\Gamma}_1 u(k) + \mathbf{\Gamma}_2 u(k+1),$$

where $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are given by (2.71). In order to put these equations in state-variable form, we must eliminate the term in $u(k+1)$. To do this, we define a new state, $\boldsymbol{\xi}(k) = \mathbf{x}(k) - \mathbf{\Gamma}_2 u(k)$. Then the equations are

$$\begin{aligned}
\boldsymbol{\xi}(k+1) &= \mathbf{x}(k+1) - \mathbf{\Gamma}_2 u(k+1) \\
&= \mathbf{\Phi x}(k) + \mathbf{\Gamma}_1 u(k) + \mathbf{\Gamma}_2 u(k+1) - \mathbf{\Gamma}_2 u(k+1), \\
\boldsymbol{\xi}(k+1) &= \mathbf{\Phi}[\boldsymbol{\xi}(k) + \mathbf{\Gamma}_2 u(k)] + \mathbf{\Gamma}_1 u(k) \\
&= \mathbf{\Phi}\boldsymbol{\xi}(k) + (\mathbf{\Phi\Gamma}_2 + \mathbf{\Gamma}_1) u(k) \\
&= \mathbf{\Phi}\boldsymbol{\xi}(k) + \mathbf{\Gamma} u(k).
\end{aligned} \tag{2.74}$$

The output equation is

$$\begin{aligned}
y(k) &= \mathbf{Hx}(k) \\
&= \mathbf{H}[\boldsymbol{\xi}(k) + \mathbf{\Gamma}_2 u(k)] \\
&= \mathbf{H}\boldsymbol{\xi}(k) + \mathbf{H\Gamma}_2 u(k) \\
&= \mathbf{H}_d\boldsymbol{\xi}(k) + J_d u(k).
\end{aligned} \tag{2.75}$$

Thus for $\ell = 0$, the state equations are given by (2.71), (2.74), and (2.75). Note especially that if $m = 0$, then $\mathbf{\Gamma}_2 = 0$, and these equations reduce to the previous model with no delay.

Our next case is $\ell = 1$. From (2.67), the equations are given by

$$\mathbf{x}(k+1) = \mathbf{\Phi x}(k) + \mathbf{\Gamma}_1 u(k-1) + \mathbf{\Gamma}_2 u(k).$$

In this case, we must eliminate $u(k-1)$ from the right-hand side, which we do by defining a new state $x_{n+1}(k) = u(k-1)$. We have thus an increased dimension of the state, and the equations are

$$\begin{bmatrix} \mathbf{x}(k+1) \\ x_{n+1}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Gamma}_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_{n+1}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_2 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = [\mathbf{H} \quad 0] \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix}. \tag{2.76}$$

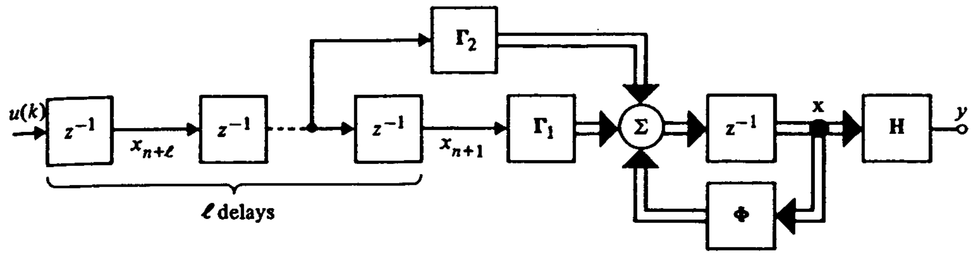


Figure 2.20 Block diagram of system with delay of more than one period. Double line indicates vector valued variables.

For our final case, we consider $\ell > 1$. In this case, the equations are

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma_1 u(k-\ell) + \Gamma_2 u(k-\ell+1),$$

and we must eliminate the past controls up to $u(k)$. To do this we introduce ℓ new variables such that

$$x_{n+1}(k) = u(k-\ell), \qquad x_{n+2}(k) = u(k-\ell+1), \ldots, x_{n+\ell}(k) = u(k-1).$$

The structure of the equations is

$$\begin{bmatrix} \mathbf{x}(k+1) \\ x_{n+1}(k+1) \\ x_{n+2}(k+1) \\ \vdots \\ x_{n+\ell}(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Gamma_1 & \Gamma_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & & & 1 \\ 0 & 0 & 0 & \vdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_{n+1}(k) \\ x_{n+2}(k) \\ \vdots \\ x_{n+\ell}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k),$$

$$y(k) = [\mathbf{H} \quad 0] \begin{bmatrix} \mathbf{x} \\ x_{n+1} \\ \vdots \\ x_{n+\ell} \end{bmatrix}. \tag{2.77}$$

This final situation is easily visualized in terms of a block diagram, as shown in Fig. 2.20.

2.4.5 Numerical Considerations in Computing Φ and Γ

The numerical considerations of these computations are centered in the approximation to the infinite sum for Ψ given by (2.72) or, for $a = T$, by (2.58).

The problem is that if $\mathbf{F}T$ is large, then $(\mathbf{F}T)^N/N!$ becomes extremely large before it becomes small, and before acceptable accuracy is realized most computer number representations will overflow, destroying the value of the computation. Källström (1973) has analyzed a technique used by Kalman and Englar (1966), which has been found effective by Moler and Van Loan (1978). The basic idea comes from (2.50) with $t_2 - t_0 = 2T$ and $t_1 - t_0 = T$, namely,

$$(e^{\mathbf{F}T})^2 = e^{\mathbf{F}T}e^{\mathbf{F}T} = e^{\mathbf{F}2T}. \tag{2.78}$$

Thus, if T is too large, we can compute the series for $T/2$ and square the result. If $T/2$ is too large, we compute the series for $T/4$, and so on, until we find a k such that $T/2^k$ is *not* too large. We need a test for deciding on the value of k. We propose to approximate the series for $\mathbf{\Psi}$, which can be written

$$\mathbf{\Psi}\left(\frac{T}{2^k}\right) = \sum_{j=0}^{N-1} \frac{[\mathbf{F}(T/2^k)]^j}{(j+1)!} + \sum_{j=N}^{\infty} \frac{(\mathbf{F}T/2^k)^j}{(j+1)!} = \hat{\mathbf{\Psi}} + \mathbf{R}.$$

We will select k, the factor that decides how much the sample period is divided down, to yield a small remainder term $\mathbf{R}$. Källström suggests that we estimate the size of $\mathbf{R}$ by the size of the first term ignored in $\hat{\mathbf{\Psi}}$, namely,

$$\hat{\mathbf{R}} \cong (\mathbf{F}T)^N/(N+1)!2^{Nk}.$$

A simpler method is to select k such that the size of $\mathbf{F}T$ divided by 2^k is less than 1. In this case, the series for $\mathbf{F}T/2^k$ will surely converge. The rule is to select k such that

$$2^k > \|\mathbf{F}T\| = \max_j \sum_{i=1}^{n} |F_{ij}|T.$$

Taking the log of both sides, we find

$$k > \log_2 \|\mathbf{F}T\|,$$

from which we select

$$k = \max(\lceil \log_2 \|\mathbf{F}T\|, 0), \tag{2.79}$$

where the symbol $\lceil x$ means the smallest integer greater than x. The maximum of this integer and zero is taken because it is possible that $\|\mathbf{F}T\|$ is

1. Select $\mathbf{F}$ and T.
2. Comment: Compute $\|\mathbf{F}T\|$.
3. $V \leftarrow \max_j\{\Sigma_i |F_{ij}|\} \times T$
4. $k \leftarrow$ smallest nonnegative integer greater than $\log_2 V$.
5. Comment: compute $\Psi(T/2^k)$.
6. $T_1 \leftarrow T/2^k$
7. $\mathbf{I} \leftarrow$ Identity
8. $\Psi \leftarrow \mathbf{I}$
9. $j \leftarrow 11$
10. If $j = 1$,
11. $\Psi \leftarrow I + \dfrac{\mathbf{F}T_1}{j}\Psi$
12. $j \leftarrow j - 1$
13. Go to step 10.
14. Comment: Now double Ψ k times.
15. If $k = 0$, stop.
16. $\Psi \leftarrow \left(\mathbf{I} + \dfrac{\mathbf{F}T}{2^{k+1}}\Psi\right)\Psi$
17. $k \leftarrow k - 1$ go to step 14.
18. Go to step 15.

Figure 2.21 Logic for a program to compute Ψ using automatic time scaling.

already so small that its log is negative, in which case we want to select $k = 0$.

Having selected k, we now have the problem of computing $\hat{\Psi}(T)$ from $\hat{\Psi}(T/2^k)$. Our original concept was based on the series for Φ, which satisfied (2.78). To obtain the suitable formula for Ψ, we use the relation between Φ and Ψ given by (2.58) as follows to obtain the "doubling" formula for Ψ:

$$\begin{aligned}
\Phi(2T) &= \Phi(T)\Phi(T), \\
\mathbf{I} + 2T\mathbf{F}\Psi(2T) &= [\mathbf{I} + T\mathbf{F}\Psi(T)][\mathbf{I} + T\mathbf{F}\Psi(T)] \\
&= \mathbf{I} + 2T\mathbf{F}\Psi(T) + T^2\mathbf{F}^2\Psi^2(T);
\end{aligned}$$

therefore

$$2T\mathbf{F}\Psi(2T) = 2T\mathbf{F}\Psi(T) + T^2\mathbf{F}^2\Psi^2(T).$$

This is equivalent to

$$\Psi(2T) = \left(\mathbf{I} + \frac{T\mathbf{F}}{2}\Psi(T)\right)\Psi(T),$$

which is the form to be used. The program logic for computing Ψ is

shown in Fig. 2.21.[19] This algorithm does not include the delay discussed in Section 2.4.4. For that, we must implement the logic shown in Fig. 2.20.[20]

2.5 SIGNAL ANALYSIS AND DYNAMIC RESPONSE

In Section 2.3 we demonstrated that if two variables are related by a linear constant difference equation, then the ratio of the z-transform of the output signal to that of the input is a function of the system equation alone, and the ratio is called the transfer function. A method for study of linear constant discrete systems is thereby indicated, consisting of the following steps:

1. Compute the transfer function of the system $H(z)$.
2. Compute the transform of the input signal, $E(z)$.
3. Form the product, $E(z)H(z)$, which is the transform of the output signal, U.
4. Invert the transform to obtain $u(kT)$.

If the system description is available in difference-equation form, and if the input signal is elementary, then the first three steps of this process require very little effort or computation. The final step, however, is tedious if done by hand, and, because we will later be preoccupied with design of transfer functions to give desirable responses, we attach great benefit to gaining intuition for the kind of response to be expected from a transform without actually inverting it. Our approach to this problem is to present a repertoire of elementary signals with known features and to learn their representation in the transform or z-domain. Thus, when given an unknown transform, we will be able, by reference to these known solutions, to infer the major features of the time-domain signal and thus to determine whether the unknown is of sufficient interest to warrant the effort of detailed time-response computation. To begin this process of attaching a connection between the time domain and the z-transform domain, we compute the transforms of a few elementary signals.

[19]See X-C2D in Table E.1

[20]See DELAY in Table E.1.

2.5.1 The Unit Pulse

We have already seen that the unit pulse is defined by[21]

$$\begin{aligned} e_1(k) &= 1 \qquad (k = 0) \\ &= 0 \qquad (k \neq 0) \\ &= \delta_k; \end{aligned}$$

therefore we have

$$E_1(z) = \sum_{-\infty}^{\infty} \delta_k z^{-k} = z^0 = 1. \tag{2.80}$$

This result is much like the continuous case, wherein the Laplace transform of the unit impulse is the constant 1.0.

The quantity $E_1(z)$ gives us an instantaneous method to relate signals to systems: To characterize the system $H(z)$, consider the signal $u(k)$, which is the unit pulse response; then $U(z) = H(z)$.

2.5.2 The Unit Step

Consider the unit step function defined by

$$\begin{aligned} e_2(k) &= 1 \qquad (k \geq 0) \\ &= 0 \qquad (k < 0) \\ &\triangleq 1(k). \end{aligned}$$

In this case, the z-transform is

$$\begin{aligned} E_2(z) &= \sum_{k=-\infty}^{\infty} e_2(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= \frac{1}{1 - z^{-1}} \qquad (|z^{-1}| < 1) \\ &= \frac{z}{z - 1} \qquad (|z| > 1). \end{aligned} \tag{2.81}$$

---

[21]We have shifted notation here to use $e(k)$ rather than e_k for the kth sample. We use subscripts to identify different signals.

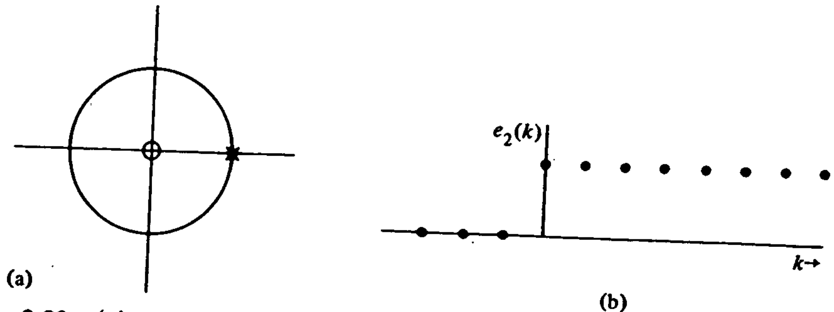


Figure 2.22 (a) Pole and zero of $E_2(z)$ in the z-plane. The unit circle is shown for reference. (b) Plot of $e_2(k)$.

Here the transform is characterized by a zero at $z = 0$ and a pole at $z = 1$. The significance of the convergence being restricted to $|z| > 1$ will be explored later when we consider the inverse transform operation. The Laplace transform of the unit step is $1/s$; we may thus keep in mind that a pole at $s = 0$ for a continuous signal corresponds in some way to a pole at $z = 1$ for discrete signals. We will explore this further later. In any event, we record that a pole at $z = 1$ with convergence outside the unit circle, $|z| = 1$, will correspond to a constant for positive time and zero for negative time.

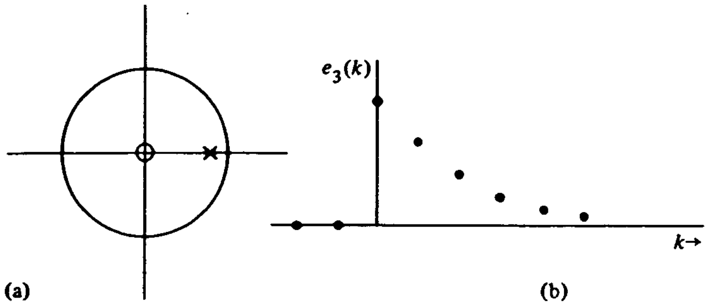
To emphasize the connection between the time domain and the z-plane, we sketch in Fig. 2.22 the z-plane with the unit circle shown and the pole of $E_2(z)$ marked $\times$ and the zero marked $\circ$. Beside the z-plane, we sketch the time plot of $e_2(k)$.

2.5.3 Exponential

The one-sided exponential in time is

$$
\begin{aligned}
e_3(k) &= r^k \qquad (k \geq 0) \\
&= 0 \qquad (k < 0),
\end{aligned}
\tag{2.82}
$$

which is the same as $r^k 1(k)$, using the symbol $1(k)$ for the unit step function.



(a) (b)

Figure 2.23 (a) Pole and zero of $E_3(z)$ in the z-plane. (b) Plot of $e_3(k)$.

Now we get

$$\begin{aligned} E_3(z) &= \sum_{k=0}^{\infty} r^k z^{-k} \\ &= \sum_{k=0}^{\infty} (rz^{-1})^k \\ &= \frac{1}{1 - rz^{-1}} \qquad (|rz^{-1}| < 1) \\ &= \frac{z}{z - r} \qquad (|z| > |r|). \end{aligned} \tag{2.83}$$

The pole of $E_3(z)$ is at $z = r$. From (2.82) we know that $e_3(k)$ grows without bound if $|r| > 1$. From (2.83) we conclude that a z-transform that converges for large z and has a real pole *outside* the circle $|z| = 1$ corresponds to a growing signal. If such a signal were the unit-pulse response of our system, such as our digital control program, we would say the program was *unstable* as we saw in (2.37). We plot in Fig. 2.23 the z-plane and the corresponding time history of $E_3(z)$ as $e_3(k)$ for the stable value, $r = 0.6$.

2.5.4 General Sinusoid

Our next example considers the modulated sinusoid $e_4(k) = [r^k \cos k\theta]1(k)$, where we assume $r > 0$. Actually, we can decompose $e_4(k)$ into the sum of two complex exponentials as

$$e_4(k) = r^k \left(\frac{e^{jk\theta} + e^{-jk\theta}}{2} \right) 1(k),$$

and because the z-transform is linear,[22] we need only compute the transform of each single complex exponential and add the results later. We thus take first

$$e_5(k) = r^k e^{jk\theta} 1(k) \tag{2.84}$$

and compute

$$\begin{aligned} E_5(z) &= \sum_{k=0}^{\infty} r^k e^{j\theta k} z^{-k} \\ &= \sum_{j=0}^{\infty} (re^{j\theta} z^{-1})^k \\ &= \frac{1}{1 - re^{j\theta} z^{-1}} \\ &= \frac{z}{z - re^{j\theta}} \qquad (|z| > r). \end{aligned} \tag{2.85}$$

The signal $e_5(k)$ grows without bound as k gets large if and only if $r > 1$, and a system with this pulse response is BIBO stable if and only if $|r| < 1$. The boundary of stability is the unit circle. To complete the argument given above for $e_4(k) = r^k \cos k\theta 1(k)$, we see immediately that the other half is found by replacing θ by $-\theta$ in (2.85),

$$\mathcal{Z}\{r^k e^{-j\theta k} 1(k)\} = \frac{z}{z - re^{-j\theta}} \qquad (|z| > r), \tag{2.86}$$

and thus that

$$\begin{aligned} E_4(z) &= \frac{1}{2}\left\{\frac{z}{z - re^{j\theta}} + \frac{z}{z - re^{-j\theta}}\right\} \\ &= \frac{z(z - r\cos\theta)}{z^2 - 2r(\cos\theta)z + r^2} \qquad (|z| > r). \end{aligned} \tag{2.87}$$

The z-plane pole-zero pattern of $E_4(z)$ and the time plot of $e_4(k)$ are shown in Fig. 2.24 for $r = 0.7$ and $\theta = 45°$.

We note in passing that if $\theta = 0$, then e_4 reduces to e_3 and, with $r = 1$, to e_2, so that three of our signals are special cases of e_4. By exploiting the

[22] We have not shown this formally. The demonstration, using the definition of linearity given above, is simple and is given in Section 2.7.

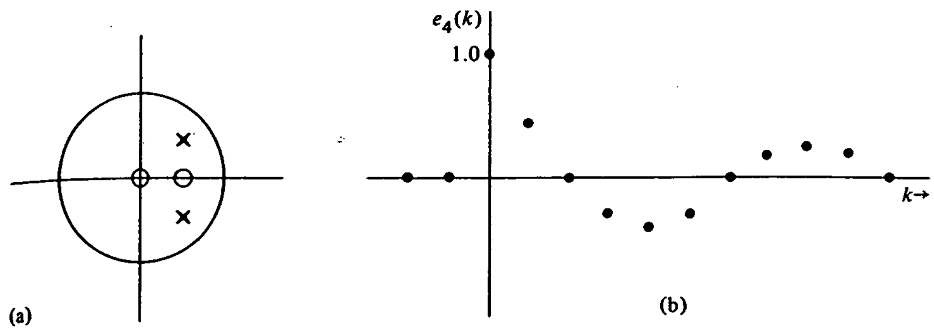


Figure 2.24 (a) Poles and zeros of $E_4(z)$ for $\theta = 45°$, $r = 0.7$ in the z-plane. (b) Plot of $e_4(k)$.

features of $E_4(z)$, we can draw a number of conclusions about the relation between pole locations in the z-plane and the time-domain signals to which the poles correspond. We collect these for later reference.

1. *The settling time of a transient, defined as the time required for the signal to decay to one percent of its maximum value, is set mainly by the value of the radius, r, of the poles.*
   - **a)** $r > 1$ corresponds to a growing signal that will not decay at all.
   - **b)** $r = 1$ corresponds to a signal with constant amplitude (which is *not* BIBO stable as a pulse response).
   - **c)** For $r < 1$, the closer r is to 0 the shorter the settling time. The corresponding system is BIBO stable. We can compute the settling time in samples, N, in terms of the pole radius, r.

pole radius, r	**response duration,** N
0.9	43
0.8	21
0.6	9
0.4	5

   - **d)** A pole at $r = 0$ corresponds to a transient of finite duration.
2. *The number of samples per oscillation of a sinusoidal signal is determined by θ.* If we require $\cos\theta k = \cos(\theta(k+N))$, we find that a period

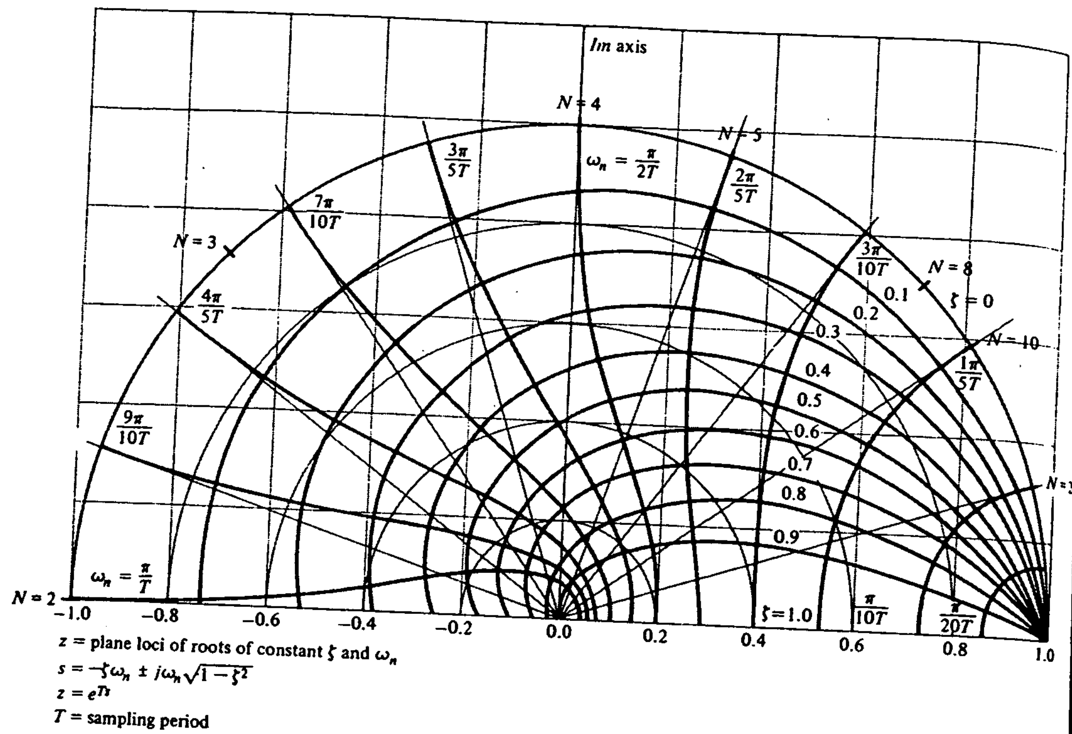


Figure 2.25 Sketch of the unit circle with angle θ marked in numbers of samples per cycle.

of 2π rad contains N samples, where

$$N = \left.\frac{2\pi}{\theta}\right|_{\text{rad}} = \left.\frac{360}{\theta}\right|_{\text{deg}} \text{ samples/cycle.}$$

For $\theta = 45°$, we have $N = 8$, and the plot of $e_4(k)$ given in Fig. 2.24(b) shows the eight samples in the first cycle very clearly. A sketch of the unit circle with several points corresponding to various numbers of samples per cycle marked is drawn in Fig. 2.25. The sampling frequency in Hertz is $1/T$, and the signal frequency is $f = 1/NT$ so that $N = f_s/f$ and $1/N$ is a *normalized* signal frequency. Since $\theta = (2\pi)/N$, θ is the normalized signal frequency in radians/sample. θ/T is the frequency in radians/second.

2.5.5 Correspondence with Continuous Signals

From the calculation of these few z-transforms, we have established that the duration of a time signal is related to the radius of the pole locations and the number of samples per cycle is related to the angle, θ. Another set of very useful relationships can be established by considering the signals to be samples from a continuous signal, $e(t)$, with Laplace transform $E(s)$. With this device we can exploit our knowledge of s-plane features by transferring them to equivalent z-plane properties. For the specific numbers represented in the illustration of e_4, we take the continuous signal

$$y(t) = e^{-at} \cos bt \, 1(t) \tag{2.88}$$

with

$$\begin{aligned} aT &= 0.3567, \\ bT &= \pi/4. \end{aligned}$$

And, taking samples one second apart ($T = 1$), we have

$$\begin{aligned} y(kT) &= (e^{-0.3567})^k \cos \frac{\pi k}{4} 1(k) \\ &= (0.7)^k \cos \frac{\pi k}{4} 1(k) \\ &= e_4(k). \end{aligned}$$

The poles of the Laplace transform of $y(t)$ (in the s-plane) are at

$$s_{1,2} = -a + jb, \; -a - jb.$$

From (2.87), the z-transform of $E_4(z)$ has poles at

$$z_{1,2} = re^{j\theta}, \; re^{-j\theta},$$

but because $y(kT)$ equals $e_4(k)$, it follows that

$$\begin{aligned} r &= e^{-aT}, \qquad \theta = bT, \\ z_{1,2} &= e^{s_1 T}, \qquad e^{s_2 T}. \end{aligned}$$

If $E(z)$ is a ratio of polynomials in z, which will be the case if $e(k)$ is generated by a linear difference equation with constant coefficients, then by partial fraction expansion, $E(z)$ can be expressed as a sum of elementary

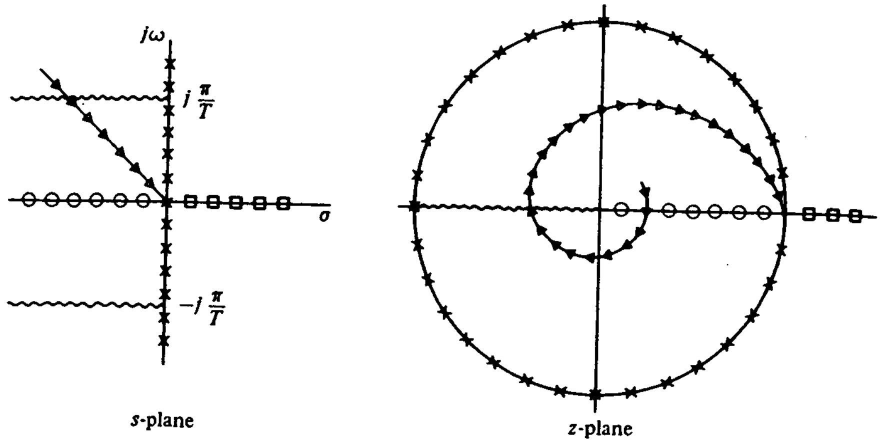


Figure 2.26 Corresponding lines in the s-plane and the z-plane according to $z = e^{sT}$.

terms like E_4 and E_3.[23] In all such cases, the discrete signal can be generated by samples from continuous signals where the relation between the s-plane poles and the corresponding z-plane poles is given by

$$z = e^{sT}. \tag{2.89}$$

If we know what it means to have a pole in a certain place in the s-plane, then (2.89) shows us where to look in the z-plane to find a representation of discrete samples having the *same time features*. It is useful to sketch several major features from the s-plane to the z-plane according to (2.89) to help fix these ideas. Such a sketch is shown in Fig. 2.26.

Each feature should be traced in the mind to obtain a good grasp of the relation. These features are given in Table 2.2. We note in passing that the map $z = e^{sT}$ of (2.89) is many-to-one. There are many values of s for each value of z. In fact, if

$$s_2 = s_1 + j\frac{2\pi}{T}N,$$

then $e^{s_1 T} = e^{s_2 T}$. The (great) significance of this fact will be explored in Chapter 3.

---

[23] Unless a pole of $E(z)$ is repeated. We have yet to compute the discrete version of a signal corresponding to a higher-order pole. The result is readily shown to be a polynomial in k multiplying $r^k e^{jk\theta}$.

Table 2.2 Descriptions of corresponding lines in s-plane and z-plane.

s-plane	Symbol	z-plane
$s = j\omega$ Real frequency axis	× × ×	$\lvert z \rvert = 1$ Unit circle
$s = \sigma \geq 0$	□□□	$z = r \geq 1$
$s = \sigma \leq 0$	○○○	$z = r,\ 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	△△△	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma, \sigma \leq 0$		$z = -r$

2.5.6 Step Response

Our eventual purpose, of course, is to design digital controls, and our interest in the relation between z-plane poles and zeros and time-domain response comes from our need to know how a proposed design will respond in a given dynamic situation. The generic dynamic test for controls is the step response, and we will conclude this discussion of discrete system dynamic response with an examination of the relationships between the pole-zero patterns of elementary systems and the corresponding step responses for a discrete transfer function from u to y of a hypothetical plant. Our attention will be restricted to the step responses of the discrete system shown in Fig. 2.27 for a selected set of values of the parameters.

Note that if $z_1 = p_1$, the members of the one pole-zero pair cancel out; and if at the same time $z_2 = r\cos(\theta)$, $a_1 = -2r\cos(\theta)$, and $a_2 = r^2$,

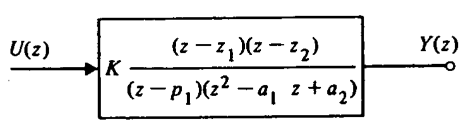


Figure 2.27 Definition of the parameters of the system whose step responses are to be catalogued.

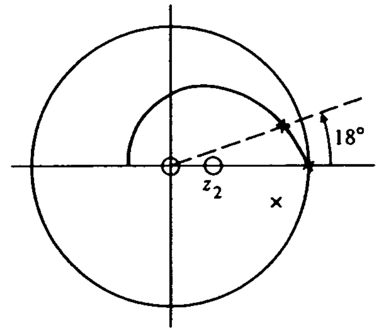


Figure 2.28 Pole-zero pattern of $Y(z)$ for the system of Fig. 2.27, with $z_1 = p_1, U(z) = z/(z-1), a_1$ and a_2 selected for $\theta = 18°$, and $\zeta = 0.5$.

the system response, $Y(z)$, to the input with transform $U(z) = 1$ (a unit pulse) is

$$Y(z) = \frac{z - r\cos\theta}{z^2 - 2r\cos\theta z + r^2}. \tag{2.90}$$

This transform, when compared with the transform $E_4(z)$ given in (2.87), is seen to be

$$Y(z) = z^{-1}E_4(z),$$

and we conclude that under these circumstances the system pulse response is a delayed version of $e_4(k)$, a typical second-order system pulse response.

For our first study we consider the effect of zero location. We let $z_1 = p_1$ and explore the effect of the (remaining) zero location, z_2, on the step-response overshoot for three sets of values of a_1 and a_2. We select a_1 and a_2 so that the poles of the system correspond to a response with damping ratio $\zeta = 0.5$ and consider values of θ of 18, 45, and 72 degrees. In every case, we will take the gain K to be such that the steady-state output value equals the step size. The situation in the z-plane is sketched in Fig. 2.28 for $\theta = 18°$. The curve for $\zeta = 0.5$ is also shown for reference. In addition to the two poles and one zero of $H(z)$, we show the pole at $z = 1$ and the zero at $z = 0$, which come from the transform of the input step, $U(z)$, given by $z/(z-1)$.

The major effect of the zero z_2 on the step response $y(k)$ is to change the percent overshoot, as can be seen from the four step responses for this case plotted in Fig. 2.29. To summarize all these data, we plot the percent overshoot versus zero location in Fig. 2.30 for $\zeta = 0.5$ and in Fig. 2.31

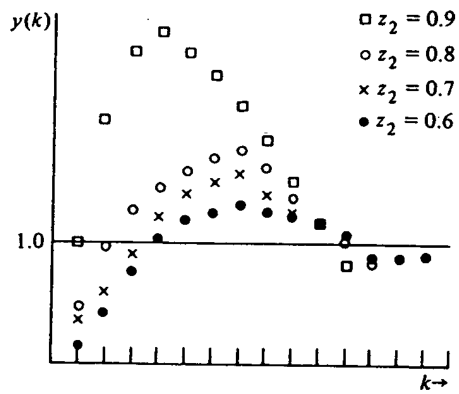


Figure 2.29 Plot of step responses for a discrete plant described by the pole-zero pattern of Fig. 2.28 for various values of z_2.

for $\zeta = 0.707$. The major feature of these plots is that the zero has very little influence when on the negative axis, but its influence is dramatic as it comes near $+1$. Also included on the plots of Fig. 2.30 are overshoot

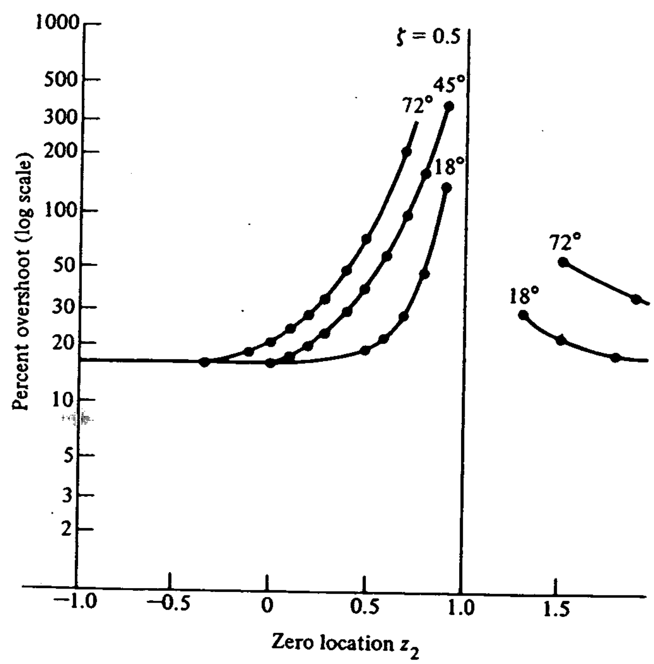


Figure 2.30 Effects of an extra zero on a discrete second-order system, $\zeta = 0.5$; $\theta = 18°$, $45°$, and $72°$.

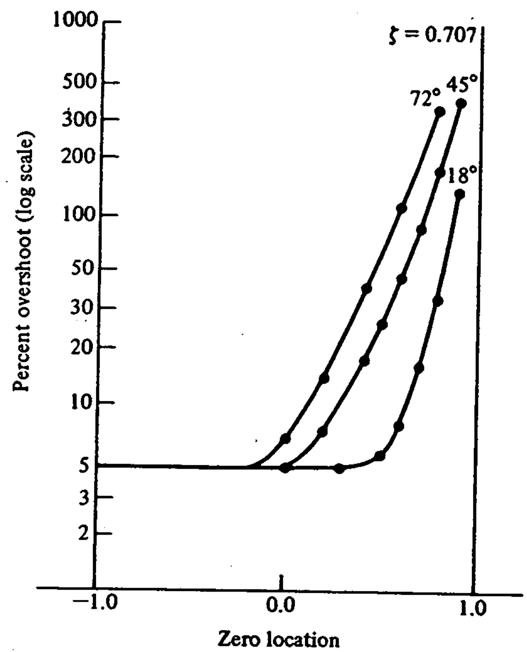


Figure 2.31 Effects of extra zero on second-order system when $\zeta = 0.707$; $\theta = 18°$, 45°, 72°. Percent overshoot versus zero location.

figures for a zero in the unstable region on the positive real axis. These responses go in the *negative* direction at first, and for the zero very near +1, the negative peak is larger than one![24]

Our second class of step responses corresponds to a study of the influence of a third pole on a basically second-order response. For this case we again consider the system of Fig. 2.27, but this time we fix $z_1 = z_2 = -1$ and let p_1 vary from near -1 to near $+1$. In this case, the major influence of the moving singularity is on the rise time of the step response. We plot this effect for $\theta = 18$, 45, and 72 degrees and $\zeta = 0.5$ on Fig. 2.32. In the figure we defined the rise time as the time required for the response to rise to 0.95, which is to 5% of its final value. We see here that the extra pole causes the rise time to get very much longer as the location of p_1 moves toward $z = +1$ and comes to dominate the response.

[24]Such systems are called nonminimum phase by Bode because the phase shift they impart to a sinusoidal input is greater than the phase of a system whose *magnitude* response is the same but that has a zero in the stable rather than the unstable region.

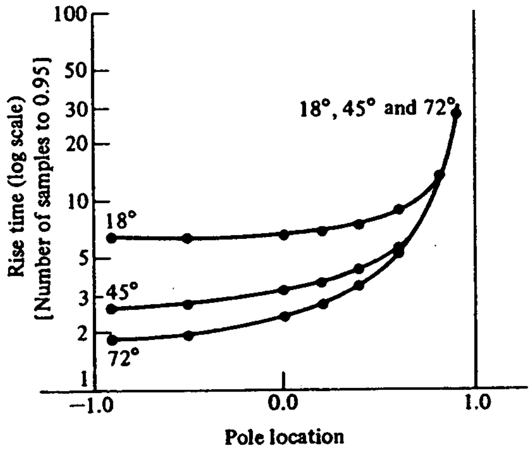


Figure 2.32 Effects of extra pole on system rise time. Two zeros at -1, one zero at ∞; $\zeta = 0.5$; $\theta = 18°, 45°, 72°$.

Our conclusions from these plots are that the addition of a pole or zero to a given system has only a small effect if the added singularities are in the range from 0 to -1. However, a zero moving toward $z = +1$ greatly increases the system overshoot. A pole placed toward $z = +1$ causes the response to slow down and thus primarily affects the rise time, which is progressively increased.

2.6 FREQUENCY RESPONSE

A very important concept in linear systems analysis is the frequency response. If a sinusoid at frequency ω_o is applied to a stable, linear, constant, continuous system, the response is a transient plus a sinusoidal steady state at the *same frequency, ω_o, as the input.* If the transfer function is written in gain-phase form as $H(j\omega) = A(\omega)e^{j\psi(\omega)}$, then the steady-state response to a unit-amplitude sinusoidal signal has amplitude $A(\omega_o)$ and phase $\psi(\omega_o)$ relative to the input signal.

We can say almost exactly the same respecting the frequency response of a stable, linear, constant, discrete system. If the system has a transfer function $H(z)$, we define its magnitude and phase for z taking on values around the unit circle by $H(e^{j\omega T}) = A(\omega T)e^{j\psi(\omega T)}$. If a unit-amplitude sinusoid is applied, then in the steady state, the response samples will be on

a sinusoid of the same frequency with amplitude $A(\omega_o T)$ and phase $\psi(\omega_o T)$. It is worthwhile going through the calculations to fix ideas on this point.

From (2.16), the discrete response transform is

$$U(z) = H(z)E(z). \tag{2.91}$$

If $e(k) = \cos(\omega_o Tk)1(k)$, then, from (2.87) with $r = 1$ and $\theta = \omega_o T$, we have

$$E(z) = \frac{1}{2}\left\{\frac{z}{z - e^{j\omega_o T}} + \frac{z}{z - e^{-j\omega_o T}}\right\}. \tag{2.92}$$

If we substitute (2.92) into (2.91), we obtain

$$U(z) = \frac{1}{2}\left\{\frac{zH(z)}{z - e^{j\omega_o T}} + \frac{zH(z)}{z - e^{-j\omega_o T}}\right\}. \tag{2.93}$$

The steady state of $u(kT)$ corresponds to the terms in the expansion of (2.93) associated with the two poles on the unit circle. If we expand $U(z)/z$ into partial fractions and multiply back by z, the steady state part can be found as

$$U_{ss}(z) = \frac{1}{2}\frac{H(e^{j\omega_o T})z}{z - e^{j\omega_o T}} + \frac{1}{2}\frac{H(e^{-j\omega_o T})z}{z - e^{-j\omega_o T}}.$$

If $H(e^{j\omega_o T}) = A(\omega_o T)e^{j\psi(\omega_o T)}$, then we have

$$U_{ss}(z) = \frac{A}{2}\frac{e^{j\psi}z}{z - e^{j\omega_o T}} + \frac{A}{2}\frac{e^{-j\psi}z}{z - e^{-j\omega_o T}}, \tag{2.94}$$

and the inverse transform of $U_{ss}(z)$ is

$$\begin{aligned} U_{ss}(kT) &= \frac{A}{2}e^{j\psi}e^{j\omega_o Tk} + \frac{A}{2}e^{-j\psi}e^{-j\omega_o Tk} \\ &= A\cos(\omega_o Tk + \psi), \end{aligned} \tag{2.95}$$

which, of course, are samples at kT instants on a sinusoid of amplitude A, phase ψ, and frequency ω_o.

We will defer the plotting of particular frequency responses until later chapters (see, for example, Figs. 4.3, 4.8, 5.16, and 5.23). However, it should be noticed here that although a sinusoid of frequency ω_o *could* be passed through the samples of (2.95), there are other continuous sinusoids of frequency $\omega_o + \ell 2\pi/T$ for integer ℓ which also pass through these points. This

is the phenomenon of aliasing, to which we will return in Chapter 3. Here, we repeat, we define the discrete frequency response of a transfer function $H(z)$ to sinusoids of frequency ω_o as $H(e^{j\omega_o T})$.

The Discrete Fourier Transform (DFT). The analysis developed above based on the z-transform is adequate for considering the theoretical frequency response of a linear, constant system or the corresponding difference equation, but it is not the best for the analysis of real-time signals as they occur in the laboratory or in other experimental situations. For the analysis of real data, we need a transform defined over a finite data record, which can be computed quickly and accurately. The required formula is that of the Discrete Fourier Transform, the DFT, and its numerical cousin, the Fast Fourier Transform, the FFT. Implementation of a version of the FFT algorithm is contained in all signal-processing software and in most computer-aided control-design software.

To understand the DFT, it is useful to consider two properties of a signal and its Fourier transform that are complements of each other: the property of being periodic and the property of being discrete. In ordinary Fourier analysis, we have a signal that is neither periodic nor discrete and its Fourier transform is also neither discrete nor periodic. If, however, the time function $f(t)$ is periodic with period T_0, then the appropriate form of the transform is the Fourier series, and the transform is defined only for the discrete frequencies $\omega = 2\pi n/T_0$. In other words, if the function in time is periodic, the function in frequency is discrete. The case where the properties are reversed is the z-transform we have just been studying. In this case, the time functions are discrete, being sampled, and the z-transform is periodic in ω; for if $z = e^{j\omega T}$, corresponding to real frequencies, then replacing $\omega = \omega + 2\pi k/T$ leaves z unchanged. We can summarize these results with the following table:

	Time	Frequency
Fourier series	periodic	discrete
z-transform	discrete	periodic

Suppose we now have a time function that is both periodic and discrete. Based on what we have seen, we would expect the transform of this function

also to be both periodic and discrete. And this is the case, which leads us to the finite discrete Fourier transform and its finite inverse. Let the time function in question be $f(kT) = f(kT + NT)$. Because the function is periodic, the transform can be defined as the finite sum

$$F\left(\frac{2\pi n}{NT}\right) = \sum_{k=0}^{N-1} f(kT)e^{-j2\pi(nkT)/(NT)}.$$

This is the same as the z-transform over one period evaluated at the discrete frequencies of a Fourier series $\omega = 2\pi n/NT$. It is standard practice to suppress all the arguments except the indices of time and frequency and write

$$F_n = \sum_{k=0}^{N-1} f_k e^{-j2\pi(nk)/N}. \tag{2.96}$$

To complete the DFT, we need the inverse transform, which, by analogy with the standard Fourier transform, we guess to be the sum

$$\sum_{n=0}^{N-1} F_n e^{j2\pi(nk)/N}.$$

If we substitute (2.96) with summing index ℓ into this, we find

$$\sum_{n=0}^{N-1} \left\{ \sum_{\ell=0}^{N-1} f_\ell e^{-j2\pi(n\ell)/N} \right\} e^{j2\pi(nk)/N}.$$

Interchanging the order of the summations gives

$$\sum_{\ell=0}^{N-1} f_\ell \left\{ \sum_{n=0}^{N-1} e^{j2\pi[n(k-\ell)]/N} \right\}.$$

The sum in the braces is a finite geometric series, which we can evaluate as follows:

$$\begin{aligned}\sum_{n=0}^{N-1} e^{j2\pi[n(k-\ell)]/N} &= \frac{1 - e^{j2\pi(k-\ell)}}{1 - e^{j2\pi(k-\ell)/N}} \\ &= \begin{cases} N & k-\ell = 0 \\ 0 & k-\ell = 1, 2, \ldots, N-1. \end{cases}\end{aligned}$$

The sum is periodic with period N. With this evaluation, we see that the sum we have been considering is Nf_k, and thus we have the inverse sum,

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{j2\pi(nk)/N}. \tag{2.97}$$

Equations (2.96) and (2.97) comprise the DFT:

$$F_n = \sum_{k=0}^{N-1} f_k e^{-j2\pi(nk)/N},$$

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{j2\pi(nk)/N}.$$

Because there are N terms in the sum in (2.96), it would appear that to compute the DFT for one frequency it will take on the order of N multiply and add operations, and to compute the DFT for all N frequencies, it would take on the order of N^2 multiply and add operations. However, several authors, especially Cooley and Tukey(1965), have showed how to take advantage of the circular nature of the exponential so that all N values of F_n can be computed with on the order of $N\log(N)$ operations if N is a power of 2. For $N = 1024$, this is a saving of a factor of 100, a very large value. Their algorithm and related schemes are called the Fast Fourier Transform, or FFT.

To use the DFT/FFT in evaluating frequency response, we suppose we have a system described by (2.91) and that the input is a sinusoid at frequency $\omega_\ell = 2\pi\ell/NT$ so that $e(kT) = A\sin(2\pi\ell kT/NT)$. We apply this input to the system *and wait until all transients have died away.* At this time, the output is given by $u(kT) = B\sin(2\pi\ell k/N + \psi)$. The DFT of $e(k)$ is

$$\begin{aligned}
E_n &= \sum_{k=0}^{N-1} A\sin\left(\frac{2\pi\ell k}{N}\right) e^{-j(2\pi nk)/N} \\
&= \sum_{k=0}^{N-1} \frac{A}{2j}\left[e^{j(2\pi\ell k)/N} - e^{-j(2\pi\ell k)/N}\right] e^{-j(2\pi nk)/N} \\
&= \begin{cases} 0, & \ell \neq n, \\ \dfrac{NA}{2j}, & \ell = n. \end{cases}
\end{aligned}$$

The DFT of the output is

$$\begin{aligned} U_n &= \sum_{k=0}^{N-1} B \sin\left(\frac{2\pi\ell k}{N} + \psi\right) e^{-j(2\pi n k)/N} \\ &= \sum_{k=0}^{N-1} \frac{B}{2j}\left[e^{j\psi}e^{j(2\pi\ell k)/N} - e^{-j\psi}e^{-j(2\pi\ell k)/N}\right] e^{-j(2\pi n k)/N} \\ &= \begin{cases} 0, & \ell \neq n, \\ \dfrac{NB}{2j}e^{j\psi}, & \ell = n. \end{cases} \end{aligned}$$

Dividing these results, we see that with sinusoidal input and output, the frequency response at the frequency $\omega = (2\pi\ell)/NT$ is given by

$$H(e^{j(2\pi\ell)/N}) = \frac{U_\ell}{E_\ell},$$

where $U_\ell = FFT(u_k)$ and $E_\ell = FFT(e_k)$, each evaluated at $n = \ell$. We will discuss in Chapter 8 the general problem of estimation of the total frequency response from experimental data using the DFT/FFT as well as other tools.

2.7 PROPERTIES OF THE z-TRANSFORM

We have used the z-transform to show that linear, constant, discrete systems can be described by a transfer function that is the z-transform of the system's unit-pulse response, and we have studied the relationship between the pole-zero patterns of transfer functions in the z-plane and the corresponding time responses. We began a table of z-transforms, and a more extensive table is given in Appendix B. In Section 2.7.1 we turn to consideration of some of the properties of the z-transform that are essential to the effective and correct use of this important tool. In Section 2.7.2 an alternate derivation of the transfer function is given.

2.7.1 z-Transform Properties

In order to make maximum use of a table of z-transforms, one must be able to use a few simple properties of the z-transform which follow directly from the definition. Some of these, such as linearity, we have already used without making a formal statement of it, and others, such as the transform of the

convolution, we have previously derived. For reference, we will demonstrate a few properties here and collect them into Appendix B for future reference. In all the properties listed below, we assume that $F_i(z) = \mathcal{Z}\{f_i(kT)\}$.

1. *Linearity:* A function $f(x)$ is linear if $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$. Applying this result to the definition of the z-transform, we find immediately that

$$\begin{aligned}\mathcal{Z}\{\alpha f_1(kT) + \beta f_2(kT)\} &= \sum_{k=-\infty}^{\infty} \{\alpha f_1(k) + \beta f_2(k)\} z^{-k} \\ &= \alpha \mathcal{Z}\{f_1(k)\} + \beta \mathcal{Z}\{f_2(k)\} \\ &= \alpha F_1(z) + \beta F_2(z).\end{aligned}$$

   Thus the z-transform is a linear function. It is the linearity of the transform that makes the partial-fraction technique work.

2. *Convolution of Time Sequences:*

$$\mathcal{Z}\left\{\sum_{l=-\infty}^{\infty} f_1(l) f_2(k-l)\right\} = F_1(z)F_2(z).$$

   We have already developed this result in connection with (2.30). It is this result with linearity that makes the transform so useful in linear-constant-system analysis because the analysis of a combination of such dynamic systems can be done by linear algebra on the transfer functions.

3. *Time Shift:*

$$\mathcal{Z}\{f(k+n)\} = z^{+n} F(z). \tag{2.98}$$

   We demonstrate this result also by direct calculation:

$$\mathcal{Z}\{f(k+n)\} = \sum_{k=-\infty}^{\infty} f(k+n) z^{-k}.$$

   If we let $k + n = j$, then

$$\begin{aligned}\mathcal{Z}\{f(k+n)\} &= \sum_{j=-\infty}^{\infty} f(j) z^{-(j-n)} \\ &= z^n F(z). \qquad \text{QED}\end{aligned}$$

This property is the essential tool in solving linear constant-coefficient difference equations by transforms. We should note here that the transform of the time shift is not the same for the one-sided transform because a shift can introduce terms with negative argument which are not included in the one-sided transform and must be treated separately. This effect causes initial conditions for the difference equation to be introduced when solution is done with the one-sided transform. See Problem 2.13.

4. *Scaling in the z-plane:*

$$\mathcal{Z}\{r^{-k}f(k)\} = F(rz). \tag{2.99}$$

By direct substitution, we obtain

$$\begin{aligned}\mathcal{Z}\{r^{-k}f(k)\} &= \sum_{k=-\infty}^{\infty} r^{-k}f(k)z^{-k} \\ &= \sum_{k=-\infty}^{\infty} f(k)(rz)^{-k} \\ &= F(rz). \qquad \text{QED}\end{aligned}$$

As an illustration of this property, we consider the z-transform of the unit step, $1(k)$, which we have computed before:

$$\mathcal{Z}\{1(k)\} = \sum_{k=0}^{\infty} z^{-k} = \frac{z}{z-1}.$$

By property 4 we have immediately that

$$\mathcal{Z}\{r^{-k}1(k)\} = \frac{rz}{rz-1} = \frac{z}{z-(1/r)}.$$

As a more general example, if we have a polynomial $a(z) = z^2 + a_1 z + a_2$ with roots $re^{\pm j\theta}$, then the scaled polynomial $\alpha^2 z^2 + a_1 \alpha z + a_2$ has roots $(r/\alpha)e^{\pm j\theta}$. This is an example of radial projection whereby the roots of a polynomial can be projected radially simply by changing the coefficients of the polynomial. The technique is sometimes used in pole-placement designs as described in Chapter 6, and sometimes used in adaptive control as described in Chapter 11.

5. *Final-Value Theorem:* If $F(z)$ converges for $|z| > 1$ and all poles of $(z-1)F(z)$ are inside the unit circle, then

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)F(z). \tag{2.100}$$

The conditions on $F(z)$ assure that the only possible pole of $F(z)$ not strictly inside the unit circle is a simple pole at $z = 1$, which is removed in $(z-1)F(z)$. Furthermore, the fact that $F(z)$ converges as the magnitude of z gets arbitrarily large ensures that $f(k)$ is zero for negative k. Therefore, all components of $f(k)$ tend to zero as k gets large, with the possible exception of the constant term due to the pole at $z = 1$. The size of this constant is given by the coefficient of $1/(z-1)$ in the partial-fraction expansion of $F(z)$, namely,

$$C = \lim_{z\to 1}(z-1)F(z).$$

However, because all other terms in $f(k)$ tend to zero, the constant C is the final value of $f(k)$, and (2.100) results. QED

As an illustration of this property, we consider the signal whose transform is given by

$$U(z) = \frac{z}{z-0.5}\frac{T}{2}\frac{z+1}{z-1}, \qquad |z| > 1.$$

Because $U(z)$ satisfies the conditions of (2.100), we have

$$\begin{aligned}
\lim_{k\to\infty} u(k) &= \lim_{z\to 1}(z-1)\frac{z}{z-0.5}\frac{T}{2}\frac{z+1}{z-1} \\
&= \lim_{z\to 1} \frac{z}{z-0.5}\frac{T}{2}(z+1) \\
&= \frac{1}{1-0.5}\frac{T}{2}(1+1) \\
&= 2T.
\end{aligned}$$

This result can be checked against the closed form for $u(k)$ given by (2.106) below.

6. *Inversion:* As with the Laplace transform, the z-transform is actually one of a pair of transforms that connect functions of time to functions of the complex variable z. The z-transform computes a function of z from

a sequence in k. (We identify the sequence number k with time in our analysis of dynamic systems, but there is nothing in the transform *per se* that requires this.) The inverse z-transform is a means to compute a sequence in k from a given function of z. We first examine two elementary schemes for inversion of a given $F(z)$ which can be used if we know beforehand that $F(z)$ is rational in z and converges as z approaches infinity. For a sequence $f(k)$, the z-transform has been defined as

$$F(z) = \sum_{k=-\infty}^{\infty} f(k)z^{-k}, \qquad r_0 < |z| < R_0. \tag{2.101}$$

If any value of $f(k)$ for negative k is nonzero, then there will be a term in (2.101) with a positive power of z. This term will be unbounded if the magnitude of z is unbounded; and thus if $F(z)$ converges as $|z|$ approaches infinity, we know that $f(k)$ is zero for $k < 0$. In this case, (2.101) is one-sided, and we can write

$$F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}, \qquad r_0 < |z|. \tag{2.102}$$

The right-hand side of (2.102) is a series expansion of $F(z)$ about infinity or about $z^{-1} = 0$. Such an expansion is especially easy if $F(z)$ is the ratio of two polynomials in z^{-1}. We need only divide the numerator by the denominator in the correct way, and when the division is done, the coefficient of z^{-k} is automatically the sequence value $f(k)$. An example we have worked out before will illustrate the process. Suppose we take our system to be the trapezoid-rule integration with transfer function given by (2.14):

$$H(z) = \frac{T}{2}\frac{z+1}{z-1}, \qquad |z| > 1.$$

We will take the input to be the geometric series represented by $e_3(k)$ with $r = 0.5$. Then we have

$$E_3(z) = \frac{z}{z-0.5}, \qquad |z| > 0.5,$$

$$
\begin{aligned}
U(z) &= E_3(z)H(z) \\
&= \frac{z}{z-0.5}\frac{T}{2}\frac{z+1}{z-1}, \qquad |z| > 1.
\end{aligned} \tag{2.103}
$$

Equation (2.103) represents the transform of the system output, $u(k)$. Keeping out the factor of $T/2$, we write $U(z)$ as a ratio of polynomials in z^{-1},

$$U(z) = \frac{T}{2}\frac{1+z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}, \tag{2.104}$$

and divide as follows:

$$
\begin{array}{r|l}
 & \frac{T}{2}[1 + 2.5z^{-1} + 3.25z^{-2} + 3.625z^{-3} + \cdots \\
1 - 1.5z^{-1} + 0.5z^{-2} & 1 + z^{-1} \\
 & \underline{1 - 1.5z^{-1} + 0.5z^{-2}} \\
 & \quad 2.5z^{-1} - 0.5z^{-2} \\
 & \quad \underline{2.5z^{-1} - 3.75z^{-2} + 1.25z^{-3}} \\
 & \qquad\qquad 3.25z^{-2} - 1.25z^{-3} \\
 & \qquad\qquad \underline{3.25z^{-2} - 4.875z^{-3} + 1.625z^{-4}} \\
 & \qquad\qquad\qquad\qquad 3.625z^{-3} - 1.625z^{-4} \\
 & \qquad\qquad\qquad\qquad \underline{3.625z^{-3} - \cdots}
\end{array}
$$

By direct comparison with $U(z) = \sum_0^\infty u(k)z^{-k}$, we conclude that

$$
\begin{aligned}
u_0 &= T/2, \\
u_1 &= (T/2)2.5, \\
u_2 &= (T/2)3.25, \\
&\vdots
\end{aligned} \tag{2.105}
$$

Clearly, the use of a computer will greatly aid the speed of this process in all but the simplest of cases. Some may prefer to use synthetic division and omit copying over all the extraneous z's in the division. The process is identical to converting $F(z)$ to the equivalent difference equation and solving for the unit-pulse response.

The second special method for the inversion of z-transforms is to decompose $F(z)$ by partial-fraction expansion and look up the components of the sequence $f(k)$ in a previously prepared table. We consider again (2.103) and expand $U(z)$ as a function of z^{-1} as follows:

$$U(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}\frac{1}{1-0.5z^{-1}} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.5z^{-1}}.$$

We multiply both sides by $1 - z^{-1}$, let $z^{-1} = 1$, and compute

$$A = \frac{T}{2}\frac{2}{0.5} = 2T.$$

Similarly, at $z^{-1} = 2$, we evaluate

$$B = \frac{T}{2}\frac{1+2}{1-2} = -\frac{3T}{2}.$$

Looking back now at e_2 and e_3, which constitute our "table" for the moment, we can copy down that

$$\begin{aligned} u_k &= Ae_2(k) + Be_3(k) \\ &= 2Te_2(k) - \frac{3T}{2}e_3(k) \\ &= \left(2T - \frac{3T}{2}\left(\frac{1}{2}\right)^k\right)1(k) \\ &= \frac{T}{2}\left[4 - \frac{3}{2^k}\right]1(k). \end{aligned} \tag{2.106}$$

Evaluation of (2.106) for $k = 0, 1, 2, \ldots$ will, naturally, give the same values for $u(k)$ as we found in (2.105). We now examine more closely the role of the region of convergence of the z-transform and present the inverse-transform integral. We begin with another example. The sequence

$$f(k) = \begin{cases} -1, & k < 0, \\ 0, & k \geq 0, \end{cases}$$

has the transform

$$F(z) = \sum_{k=-\infty}^{-1} -z^{-k} = -\left[\sum_{0}^{\infty} z^k - 1\right]$$
$$= \frac{z}{z-1}, \qquad |z| < 1.$$

This transform is exactly the same as the transform of the unit step $1(k)$, (2.81), except that this transform converges *inside* the unit circle and the transform of the $1(k)$ converges outside the unit circle. Knowledge of the region of convergence is obviously essential to the proper inversion of the transform to obtain the time sequence. The inverse z-transform is the closed, complex integral[25]

$$f(k) = \frac{1}{2\pi j} \oint F(z) z^k \frac{dz}{z}, \tag{2.107}$$

where the contour is a circle in the region of convergence of $F(z)$. To demonstrate the correctness of the integral and to use it to compute inverses it is useful to apply Cauchy's residue calculus [see Churchill and Brown (1984)]. Cauchy's result is that a closed integral of a function of z which is analytic on and inside a closed contour C except at a finite number of isolated singularities z_i is given by

$$\frac{1}{2\pi j} \oint_C F(z)\, dz = \sum_i \mathrm{Res}(z_i). \tag{2.108}$$

In (2.108), $\mathrm{Res}(z_i)$ means the residue of $F(z)$ at the singularity at z_i. We will be considering only rational functions, and these have only poles as singularities. If $F(z)$ has a pole of order n at z_1, then $(z - z_1)^n F(z)$ is regular at z_1 and can be expanded in a Taylor series near z_1 as

$$\begin{aligned}(z - z_1)^n F(z) = &A_{-n} + A_{-n+1}(z - z_1) + \cdots + A_{-1}(z - z_1)^{n-1} \\ &+ A_0(z - z_1)^n + \cdots .\end{aligned} \tag{2.109}$$

The residue of $F(z)$ at z_1 is A_{-1}.

---

[25]If it is known that $f(k)$ is causal, that is, $f(k) = 0$ for $k < 0$, then the region of convergence is outside the smallest circle that contains all the poles of $F(z)$ for rational transforms. It is this property that permits inversion by partial-fraction expansion and long division.

First we will use Cauchy's formula to verify (2.108). If $F(z)$ is the z-transform of $f(k)$, then we write

$$\mathcal{I} = \frac{1}{2\pi j} \oint \sum_{l=-\infty}^{\infty} f(l) z^{-l} z^{k} \frac{dz}{z}.$$

We assume that the series for $F(z)$ converges uniformly on the contour of integration, so the series can be integrated term by term. Thus we have

$$\mathcal{I} = \frac{1}{2\pi j} \sum_{l=-\infty}^{\infty} f(l) \oint z^{k-l} \frac{dz}{z}.$$

The argument of the integral has no pole inside the contour if $k - l \geq 1$, and it has zero residue at the pole at $z = 0$ if $k - l < 0$. Only if $k = l$ does the integral have a residue, and that is 1. By (2.108), the integral is zero if $k \neq l$ and is $2\pi j$ if $k = l$. Thus $\mathcal{I} = f(k)$, which demonstrates (2.107).

To illustrate the use of (2.108) to compute the inverse of a z-transform, we will use the function $z/(z-1)$ and consider first the case of convergence for $|z| > 1$ and second the case of convergence for $|z| < 1$. For the first case,

$$f_1(k) = \frac{1}{2\pi j} \oint_{|z|=R>1} \frac{z}{z-1} z^{k} \frac{dz}{z}, \tag{2.110}$$

where the contour is a circle of radius greater than 1. Suppose $k < 0$. In this case, the argument of the integral has two poles inside the contour: one at $z = 1$ with residue

$$\lim_{z \to 1} (z-1) \frac{z^k}{z-1} = 1,$$

and one pole at $z = 0$ with residue found as in (2.109)(if $k < 0$, then z^{-k} removes the pole):

$$\begin{aligned} z^{-k} \frac{z^k}{z-1} &= -\frac{1}{1-z} \\ &= -(1 + z + z^2 + \cdots + z^{-k-1} + \cdots). \end{aligned}$$

The residue is thus -1 for all k, and the sum of the residues is zero, and

$$f_1(k) = 0, \qquad k < 0. \tag{2.111}$$

For $k \geq 0$, the argument of the integral in (2.110) has only the pole at $z = 1$ with residue 1. Thus

$$f_1(k) = 1, \qquad k \geq 0. \tag{2.112}$$

Equations (2.108) and (2.109) correspond to the unit-step function, as they should. We would write the inverse transform symbolically $\mathcal{Z}^{-1}\{.\}$ as, in this case,

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = 1(k) \tag{2.113}$$

when $z/(z-1)$ converges for $|z| > 1$.

If, on the other hand, convergence is inside the unit circle, then for $k \geq 0$, there are no poles of the integrand contained in the contour, and

$$f_2(k) = 0, \qquad k \geq 0.$$

At $k < 0$, there is a pole at the origin of z, and as before, the residue is equal to -1 there, so

$$f_2(k) = -1, \qquad k < 0.$$

In symbols, corresponding to (2.113), we have

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = 1(k) - 1$$

when $z/(z-1)$ converges for $|z| < 1$.

Although, as we have just seen, the inverse integral can be used to compute an expression for a sequence to which a transform corresponds, a more effective use of the integral is in more general manipulations. We consider one such case that will be of some interest later. First, we consider an expression for the transform of a product of two sequences. Suppose we have

$$f_3(k) = f_1(k)f_2(k),$$

and f_1 and f_2 are such that the transform of the product exists. An expression for $F_3(z)$ in terms of $F_1(z)$ and $F_2(z)$ can be developed as follows. By definition

$$F_3(z) = \sum_{k=-\infty}^{\infty} f_1(k)f_2(k)z^{-k}.$$

From the inversion integral, (2.107), we can replace $f_2(k)$ by an integral:

$$F_3(z) = \sum_{k=-\infty}^{\infty} f_1(k) z^{-k} \frac{1}{2\pi j} \oint_{C_2} F_2(\zeta)\zeta^{k} \frac{d\zeta}{\zeta}.$$

We assume that we can find a region where we can exchange the summation with the integration. The contour will be called C_3 in this case:

$$F_3(z) = \frac{1}{2\pi j} \oint_{C_3} F_2(\zeta) \sum_{k=-\infty}^{\infty} f_1(k) \left(\frac{z}{\zeta}\right)^{-k} \frac{d\zeta}{\zeta}.$$

The sum can now be recognized as $F_1(z/\zeta)$ and, when we substitute this,

$$F_3(z) = \frac{1}{2\pi j} \oint_{C_3} F_2(\zeta) F_1\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}, \tag{2.114}$$

the contour C_3 must be in the overlap of the convergence regions of $F_2(\zeta)$ and $F_1(z/\zeta)$. Then $F_3(z)$ will converge for the range of values of z for which C_3 can be found.

If we let $f_1 = f_2$ and $z = 1$ in (2.114), we have the discrete version of Parseval's theorem, where convergence is on the unit circle:

$$F_3(1) = \sum_{k=-\infty}^{\infty} f_1^2 = \frac{1}{2\pi j} \oint_{C} F_1(\zeta) F_1\left(\frac{1}{\zeta}\right) \frac{d\zeta}{\zeta}. \tag{2.115}$$

This particular theorem shows how we can compute the sum of squares of a time sequence by evaluating a complex integral in the z-domain. The result is useful in the design of systems by least squares.

2.7.2 Another Derivation of the Transfer Function

Let $\mathcal{D}$ be a discrete system which maps an input sequence, $\{e(k)\}$, into an output sequence $\{u(k)\}$.[26] Then, expressing this as an operator on $e(k)$, we have

$$u(k) = \mathcal{D}\{e(k)\}.$$

If $\mathcal{D}$ is linear, then

$$\mathcal{D}\{\alpha e_1(k) + \beta e_2(k)\} = \alpha\mathcal{D}\{e_1(k)\} + \beta\mathcal{D}\{e_2(k)\}. \tag{2.116}$$

[26]This derivation was suggested by L. A. Zadeh in 1952 at Columbia University.

If the system is constant, a shift in $e(k)$ to $e(k+j)$ must result in no other effects but a shift in the response, u. We write

$$\mathcal{D}\{e(k+j)\} = u(k+j) \qquad \text{for all } j \tag{2.117}$$

if

$$\mathcal{D}\{e(k)\} = u(k).$$

Theorem. If $\mathcal{D}$ is linear and constant and is given an input z^k for a value of z for which the output is finite at time k, then the output will be of the form $H(z)z^k$.

Proof. In general, if $e(k) = z^k$, then an arbitrary finite response can be written

$$u(k) = H(z,k)z^k.$$

Consider $e_2(k) = z^{k+j} = z^j z^k$ for some fixed j. From (2.116), if we let $\alpha = z^j$, it must follow that

$$\begin{aligned} u_2 &= z^j u(k) \\ &= z^j H(z,k)z^k \\ &= H(z,k)z^{k+j}. \end{aligned} \tag{2.118}$$

From (2.117), we must have

$$\begin{aligned} u_2(k) &= u(k+j) \\ &= H(z, j+k)z^{k+j} \qquad \text{for all } j. \end{aligned} \tag{2.119}$$

From a comparison of (2.118) and (2.119), it follows that

$$H(z,k) = H(z,k+j) \qquad \text{for all } j;$$

that is, H does not depend on the second argument and can be written $H(z)$. Thus for the elemental signal $e(k) = z^k$, we have a solution $u(k)$ of the same (exponential) shape but modulated by a ratio $H(z)$, $u(k) = H(z)z^k$.

Can we represent a general signal as a *linear sum* (integral) of such elements? We can, by the inverse integral derived above, as follows:

$$e(k) = \frac{1}{2\pi j} \oint E(z) z^k \frac{dz}{z}, \tag{2.120}$$

where

$$E(z) = \sum_{-\infty}^{\infty} e(k) z^{-k}, \qquad r < |z| < R, \tag{2.121}$$

for signals with $r < R$ for which (2.121) converges. We call $E(z)$ the z-transform of $e(k)$, and the (closed) path of integration is in the annular region of convergence of (2.121). If $e(k) = 0$, $k < 0$, then $R \to \infty$, and this region is the whole z-plane *outside* a circle of finite radius.

The consequences of linearity are that the response to a sum of signals is the sum of the responses as given in (2.116). Although (2.120) is the limit of a sum, the result still holds, and we can write

$$u(k) = \frac{1}{2\pi j} \oint E(z)[\text{response to } z^k] \frac{dz}{z},$$

but, by the theorem, the response to z^k is $H(z)z^k$. Therefore we can write

$$\begin{aligned} u(k) &= \frac{1}{2\pi j} \oint E(z)[H(z)z^k] \frac{dz}{z}. \\ &= \frac{1}{2\pi j} \oint H(z)E(z) z^k \frac{dz}{z}. \end{aligned} \tag{2.122}$$

We can define $U(z) = H(z)E(z)$ by comparison with (2.120) and note that

$$U(z) = \sum_{k=-\infty}^{\infty} u(k) z^{-k} = H(z)E(z). \tag{2.123}$$

Thus $H(z)$ is the *transfer function*, which is the ratio of the transforms of $e(k)$ and $u(k)$ as well as the amplitude response to inputs of the form z^k.

This derivation begins with linearity and stationarity and derives the z-transform as the natural tool of analysis from the fact that input signals in the form z^k produce an output that has the same shape.[27] It is somewhat

[27] Because z^k is unchanged in shape by passage through the linear constant system, we say that z^k is an eigenfunction of such systems.

more satisfying to derive the necessary transform than to start with the transform and see what systems it is good for. Better to start with the problem and find a tool than start with a tool and look for a problem. Unfortunately, the direct approach requires extensive use of the inversion integral and more sophisticated analysis to develop the main result, which is (2.123). Chacun à son goût.

2.8 SUMMARY

In this chapter we have shown how systems described by linear difference equations with constant coefficients can be described by transfer functions if the signals are represented by z-transforms. The transfer function was shown to be the z-transform of the unit-pulse response of the system; and, furthermore, the system output was shown to be the convolution of the input with the unit-pulse response. Using this result, we showed a condition for the Bounded-Input-Bounded-Output stability of a linear, constant system. The test developed by Jury for a polynomial to have all roots inside the unit circle was introduced to provide a stability test. We introduced the observer and the control canonical forms for transfer functions and gave rules for block-diagram reduction of transfer functions. We also introduced the state descriptions of these canonical forms and showed how to derive the matrices of the dynamic system in state form. We then derived the discrete transform for a sampled-data system both by transform and by state-space methods. The latter are especially well suited for computer implementation.

We studied the dynamic response of discrete systems, including especially the step response of a second-order system. The effects of the location of a zero and of a third pole were plotted, largely for future reference in design.

Several of the properties of the z-transform were demonstrated, and the calculation of the inverse of a z-transform was presented by long division, by partial-fraction expansion, and by evaluation of the inverse transform integral.

PROBLEMS AND EXERCISES

2.1 Check the following for stability:

a) $u(k) = 0.5u(k-1) - 0.3u(k-2)$

b) $u(k) = 1.6u(k-1) - u(k-2)$

c) $u(k) = 0.8u(k-1) + 0.4u(k-2)$

2.2 **a)** Derive the difference equation corresponding to the approximation of integration found by fitting a parabola to the points e_{k-2}, e_{k-1}, e_k and taking the area under this parabola between $t = kT - T$ and $t = kT$ as the approximation to the integral of $e(t)$ over this range.

b) Find the transfer function of the resulting discrete system and plot the poles and zeros in the z-plane.

2.3 Verify that the transfer function of the system of Fig. 2.8(c) is given by the same $H(z)$ as the system of Fig. 2.9(c).

2.4 **a)** Compute and plot the unit-pulse response of the system derived in Problem 2.2.

b) Is this system BIBO stable?

2.5 Consider the difference equation

$$u(k+2) = 0.25u(k).$$

a) Assume a solution $u(k) = A_i z^k$ and find the characteristic equation in z.

b) Find the characteristic roots z_1 and z_2 and decide if the equation solutions are stable or unstable.

c) Assume a general solution of the form

$$u(k) = A_1 z_1{}^k + A_2 z_2{}^k$$

and find A_1 and A_2 to match the initial conditions $u(0) = 0, u(1) = 1$.

d) Repeat parts (a), (b), and (c) for the equation

$$u(k+2) = -0.25u(k)$$

e) Repeat parts (a), (b), and (c) for the equation

$$u(k+2) = u(k+1) - 0.5u(k).$$

2.6 Show that the characteristic equation

$$z^2 - 2r\cos(\theta)z + r^2$$

has the roots

$$z_{1,2} = re^{\pm j\theta}.$$

2.7 **a)** Use the method of block-diagram reduction, applying Figs 2.5, 2.6, and 2.7 to compute the transfer function of Fig 2.8(c).

b) Repeat part (a) for the diagram of Fig. 2.9(c).

2.8 Apply Jury's test to determine if the following characteristic equations have any roots outside the unit circle.

a) $z^2 + 0.25$

b) $z^3 - 1.1z^2 + 0.01z + 0.405$

c) $z^3 - 3.6z^2 + 4z - 1.6$

2.9 Compute by hand and table look-up the discrete transfer function if the $G(s)$ in Fig. 2.13 is

a) $\dfrac{K}{s}$ b) $\dfrac{3}{s(s+3)}$

c) $\dfrac{3}{(s+1)(s+3)}$ d) $\dfrac{(s+1)}{s^2}$

e) $\dfrac{e^{sT/2}}{s^2}$ f) $\dfrac{(1-s)}{s^2}$

g) $\dfrac{3e^{-1.5Ts}}{(s+1)(s+3)}$

h) Repeat the calculation of these discrete transfer functions using a CAD tool. Compute for the sampling period $T = 0.05$ and $T = 0.5$ and plot the location of the poles and zeros in the z-plane.

2.10 Use a CAD tool to compute the discrete transfer function if the $G(s)$ in Fig. 2.13 is

a) the two-mass system with the noncolocated actuator and sensor of (1.4) with sampling periods $T = 0.02$ and $T = 0.1$. Plot the zeros and poles of the results in the z-plane.

b) the two-mass system with the colocated actuator and sensor given by (1.5). Use $T = 0.02$ and $T = 0.1$. Plot the zeros and poles of the results in the z-plane.

c) the two-input–two-output tape drive described by (1.6). In this case use sampling period $T = 0.1$ and $T = 0.5$.

2.11 Consider the system described by the transfer function

$$\frac{Y}{U} = G(s) = \frac{3}{(s+1)(s+3)}.$$

a) Draw the block diagram corresponding to this system in control canonical form, define the state vector, and give the corresponding description matrices $\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}$.

b) Write $G(s)$ in partial fractions and draw the corresponding parallel block diagram with each component part in control canonical form. Define the state ξ and give the corresponding state description matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$.

c) By finding the transfer functions X_1/U and X_2/U of part (a) in partial fraction form, express x_1 and x_2 in terms of ξ_1 and ξ_2. Write these relations as the two-by-two transformation $\mathbf{T}$ such that $\mathbf{x} = \mathbf{T}\xi$.

d) Verify that the matrices you have found are related by the formulas

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T},$$

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{G},$$

$$\mathbf{C} = \mathbf{H}\mathbf{T},$$

$$\mathbf{D} = \mathbf{J}.$$

2.12 The first-order system $(z-\alpha)/(1-\alpha)z$ has a zero at $z = \alpha$.

a) Plot the step response for this system for $\alpha = 0.8, 0.9, 1.1, 1.2, 2$.

b) Plot the overshoot of this system on the same coordinates as those appearing in Fig. 2.30 for $-1 < \alpha < 1$.

c) In what way is the step response of this system unusual for $\alpha > 1$?

2.13 The one-sided z-transform is defined as

$$F(z) = \sum_{0}^{\infty} f(k) z^{-k}.$$

a) Show that the one-sided transform of $f(k+1)$ is

$$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0).$$

b) Use the one-sided transform to solve for the transforms of the Fibonacci numbers by writing (2.4) as $u_{k+2} = u_{k+1} + u_k$. Let $u_0 = u_1 = 1$. [You will need to compute the transform of $f(k+2)$.]

c) Compute the location of the poles of the transform of the Fibonacci numbers.

d) Compute the inverse transform of the numbers.

e) Show that if u_k is the kth Fibonacci number, then the ratio u_{k+1}/u_k will go to $(1+\sqrt{5})/2$, the golden ratio of the Greeks.

f) Show that if we add a forcing term, $e(k)$, to (2.4) we can generate the Fibonacci numbers by a system that can be analyzed by the two-sided transform; i.e., let $u_k = u_{k-1} + u_{k-2} + e_k$ and let $e_k = \delta_0(k)$[$\delta_0(k) = 1$ at $k = 0$ and zero elsewhere]. Take the two-sided transform and show that the same $U(z)$ results as in part (b).

2.14 Substitute $u = Az^k$ and $e = Bz^k$ into (2.2) and (2.7) and show that the transfer functions, (2.15) and (2.14), can be found in this way.

2.15 Consider the transfer function

$$H(z) = \frac{(z+1)(z^2 - 1.3z + 0.81)}{(z^2 - 1.2z + 0.5)(z^2 - 1.4z + 0.81)}.$$

Draw a *cascade* realization, using observer canonical forms for second-order blocks and in such a way that the coefficients as shown in $H(z)$ above are the parameters of the block diagram.

2.16 **a)** Write the $H(z)$ of Exercise 2.15 in partial fractions in two terms of second order each, and draw a *parallel* realization, using the observer canonical form for each block and showing the coefficients of the partial-fraction expansion as the parameters of the realization.

b) Suppose the two factors in the denominator of $H(z)$ were identical (say we change the 1.4 to 1.2 and the 0.81 to 0.5). What would the parallel realization be in this case?

2.17 Show that the observer canonical form of the system equations shown in Fig. 2.9 can be written in the state-space form as given by (2.26).

2.18 Draw out each block of Fig. 2.10 in (a) control and (b) observer canonical form. Write out the state-description matrices in each case.

2.19 For a second-order system with damping ratio 0.5 and poles at an angle in the z-plane of $\theta = 30^\circ$, what percent overshoot to a step would you expect if the system had a zero at $z_2 = 0.6$?

2.20 Consider a signal with the transform (which converges for $|z| > 2$)

$$U(z) = \frac{z}{(z-1)(z-2)}.$$

a) What value is given by the formula (final-value theorem) of (2.100) applied to this $U(z)$?

b) Find the final value of $u(k)$ by taking the inverse transform of $U(z)$, using partial-fraction expansion and the tables.

c) Explain why the two results of (a) and (b) differ.

2.21 **a)** Find the z-transform and be sure to give the region of convergence for the signal

$$u(k) = r^{+|k|}, \qquad r < 1.$$

[*Hint:* Write u as the sum of two functions, one for $k \geq 0$ and one for $k < 0$, find the individual transforms, and determine values of z for which *both* terms converge.]

b) If a rational function $U(z)$ is known to converge on the unit circle $|z| = 1$, show how partial-fraction expansion can be used to compute the inverse transform. Apply your result to the transform you found in part (a).

2.22 Compute the inverse transform, $f(k)$, for each of the following transforms:

a) $F(z) = \dfrac{1}{1 + z^{-2}}, \qquad |z| > 1;$

b) $F(z) = \dfrac{z(z-1)}{z^2 - 1.25z + 0.25}, \qquad |z| > 1;$

c) $F(z) = \dfrac{z}{z^2 - 2z + 1}, \qquad |z| > 1;$

d) $F(z) = \dfrac{z}{(z - \frac{1}{2})(z - 2)}, \qquad 1/2 < |z| < 2.$

2.23 Use the z-transform to solve the difference equation

$$y(k) - 3y(k-1) + 2y(k-2) = 2u(k-1) - 2u(k-2),$$

$$u(k) = \begin{cases} k, & k \geq 0, \\ 0, & k < 0, \end{cases}$$

$$y(k) = 0, \qquad k < 0.$$

CHAPTEF

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3.1 INTR

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CHAPTER 3

Sampled-Data Systems

3.1 INTRODUCTION

The use of digital logic or digital computers to calculate a control action for a continuous, dynamic system introduces the fundamental operation of sampling. Samples are taken from the continuous, physical signals such as position, velocity, or temperature, and these samples are used in the computer to calculate the controls to be applied. Such digital controls are hybrids, where discrete signals appear in some places and continuous signals occur in other parts. Such systems are called *sampled-data systems* because some continuous data are sampled before being used. In many ways, the analysis of a purely continuous system or of a purely discrete system is simpler than is that of the hybrid case. However, in digital control much of the processing is done via digital logic on discrete signals, but the origin of the signals is in the continuous world, and the destination of our computed outputs is there as well. Thus the role of sampling and the conversion from continuous to discrete and back from discrete to continuous are very important to understanding digital control, and we must study the process of sampling and how to make mathematical models of analog-to-digital conversion because our real interest is in the hybrid, sampled-data case. This analysis will require some careful treatment via the Fourier transform, but the effort will be well rewarded with the understanding it will bring to later systems.

In this chapter, we introduce the analysis of the sampling process and describe both a time domain and a frequency domain representation. We also describe the companion process, that of sample extrapolation or holding to recover a continuous time signal from its samples. As part of this analysis, we show that, because a sampled-data system is made to be time

varying by the introduction of sampling, it is not possible to describe such systems exactly by a transfer function. However, after sampling and holding, a continuous signal is recovered, and we can approximate the response of a sample and hold to a sinusoid by fitting another sinusoid of the same frequency to the complete response. We will show how to compute this best-fit sinusoidal response analytically and experimentally and thus have a good approximation for a transfer function. For those familiar with the idea, our approach is equivalent to the use of the "describing function" that is used to approximate a transfer function for simple nonlinear systems. This concept will be studied in Chapter 11.

Once the operations of sampling and holding are understood, we will show that we can always represent the relationship between the samples of the input and the samples of the output of a linear constant system by a *discrete* transfer function. Thus if we are willing to focus on the samples only, the entire power of linear, constant system theory is available to us. In fact, the analysis of discrete linear systems is in many ways simpler than the analysis of continuous linear systems in the way that subtraction is simpler than differentiation. We will see this when in the next chapter we use the z-transform to analyze difference equations; however, in the digital control of continuous dynamical systems we must understand the transitions from continuous to discrete and back again from discrete to continuous signals from the start. This is what we do in this chapter for the most elementary operations: sampling and holding.

3.2 ANALYSIS OF THE SAMPLE AND HOLD

To get samples of a physical signal such as a position or a velocity into digital form, we typically have a sensor that produces a voltage proportional to the physical variable and an analog-to-digital converter, commonly called an A/D converter or ADC, that transforms the voltage into a digital number. The physical conversion always takes a finite time, and in many instances this time is significant with respect to the sample time of the controls and with respect to the rate of change of the signal being sampled. In order to give the computer an accurate representation of the signal exactly at the sampling instants kT, the A/D converter is often preceded by a circuit called the Sample-and-Hold Circuit or SHC. A simple electronic schematic is sketched in Fig. 3.1, where the switch, S, is an electronic device driven by simple logic from a clock. Its operation is described below.

With the switch, S, in position 1, the amplifier output $v_{\text{out}}(t)$ follows or tracks the input voltage $v_{\text{in}}(t)$ through the transfer function $1/(RCs+1)$. The

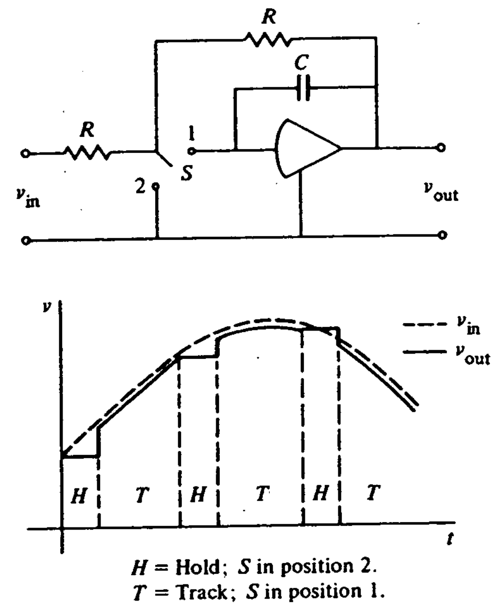
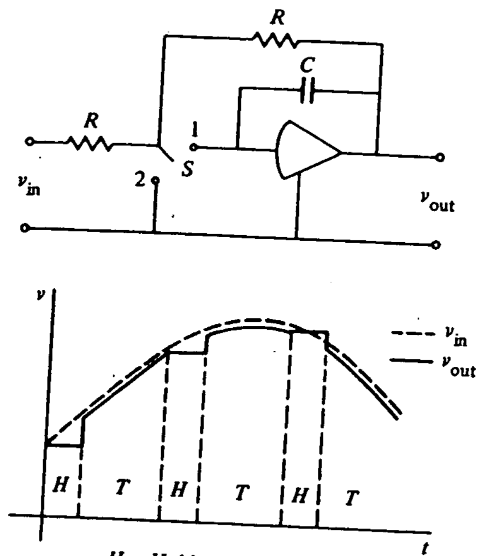


Figure 3.1 Analog-to-digital converter with sample and hold.

circuit bandwidth of the SHC, $1/RC$, is selected to be high compared to the input signal bandwidth. Typical values are $R = 1000$ ohms, $C = 30 \times 10^{-12}$ farads for a bandwidth of $f = 1/2\pi RC = 5.3$ MHz. During this "tracking time," the ADC is turned off and ignores v_{out}. When a sample is to be taken at $t = kT$ the switch S is set to position 2 and the capacitor C holds the output of the operational amplifier frozen from that time at $v_{\text{out}}(kT)$. The ADC is now signaled to begin conversion and has the constant input from the SHC to work on, so the resulting digital number is a true representation of the input voltage at the sample time. When the conversion is completed, the digital number is presented to the digital computer at which time the calculations based on this sample value can begin. The SHC switch is now moved to position 1, and the circuit is again tracking, waiting for the next command to freeze a sample. For example, the conversion time of the Burr-Brown ADC803 is 1.5 microseconds for 12 bits of accuracy. The SHC needs only to hold the voltage for this short time in order for the conversion to be completed before it is started tracking again. The value taken is held inside the computer for the entire sampling period of the system, so the combination of the electronic SHC plus the ADC operate as a sample-and-



H = Hold; S in position 2.
T = Track; S in position 1.

Figure 3.1 Analog-to-digital converter with sample and hold.

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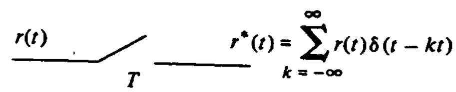

Figure 3.2 The sampler.

hold for the sampling period, T seconds, which may be many milliseconds long.

For the purpose of the analysis we separate the sample and hold into two mathematical operations: a sampling operation represented as impulse modulation and a hold operation represented as a linear filter. The symbol or schematic of the ideal sampler is shown in Fig. 3.2; its role is to give a mathematical representation of the process of taking periodic samples from $r(t)$ to produce $r(kT)$ and to do this in such a way that we can include the sampled signals in the analysis of continuous signals using the Laplace transform.[1] The technique is to use *impulse modulation* as the mathematical representation of sampling. Thus, from Fig. 3.2, we picture the output of the sampler as a string of impulses,

$$r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT). \tag{3.1}$$

The impulse can be visualized as the limit of a pulse of unit area that has growing amplitude and shrinking duration. The essential property of the impulse is the sifting property that

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{3.2}$$

for all functions f that are continuous at a. The integral of the impulse is the unit step

$$\int_{-\infty}^{t} \delta(\tau)d\tau = 1(t), \tag{3.3}$$

and the Laplace transform of the unit impulse is one, because

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(\tau)e^{-s\tau}d\tau = 1. \tag{3.4}$$

[1]We assume that the reader has some familiarity with Fourier and Laplace transform analysis. For a general reference, see Bracewell (1978).

Using these properties we can see that $r^*(t)$, defined in (3.1), depends only on the discrete sample values $r(kT)$. The Laplace transform of $r^*(t)$ can be computed as follows:

$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r^*(\tau)e^{-s\tau}\,d\tau.$$

If we substitute (3.1) for $r^*(t)$, we get

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(\tau)\delta(\tau - kT)e^{-s\tau}\,d\tau,$$

and now, exchanging integration and summation and using (3.2), we have

$$R^*(s) = \sum_{k=-\infty}^{\infty} r(kT)e^{-skT}. \tag{3.5}$$

The notation $R^*(s)$ is used to symbolize the (Laplace) transform of $r^*(t)$, the sampled or impulse-modulated $r(t)$.[2]

Having a model of the sampling operation as impulse modulation, to complete the description of the physical sample-and-hold we need to model

---

[2] It will be necessary, from time to time, to consider sampling a signal that is not continuous. The only case we will consider will be equivalent to applying a step function, $1(t)$, to a sampler. For the purposes of this book we will define the unit step to be continuous from the right and assume that the impulse, $\delta(t)$, picks up the full value of unity. By this convention and (3.1) we compute

$$1^*(t) = \sum_{k=0}^{\infty} \delta(t - kT), \tag{a}$$

and, using (3.2), we obtain

$$\mathcal{L}\{1^*(t)\} = 1/(1 - e^{-Ts}). \tag{b}$$

The reader should be warned that the Fourier integral converges to the *average* value of a function at a discontinuity and not the value approached from the right as we assume. Because our use of the transform theory is elementary and the convenience of equation (b) above is substantial, we have selected the continuous-from-the-right convention. In case of doubt, the discontinuous term should be separated and treated by special analysis, perhaps in the time domain.

the hold operation, which will take the impulses that are produced by the mathematical sampler and produce the piecewise constant output of the device. Typical signals are sketched on Fig. 3.3. Once the samples are taken, as represented by $r^*(t)$ in (3.1), the hold is defined as the means whereby these impulses are extrapolated to the piecewise constant signal r_h, defined as

$$r_h(t) = r(kT) \qquad kT \le t < kT + T. \tag{3.6}$$

Because r_h is composed of zero-order polynomials passed through the samples of $r(kT)$, this hold operation is called the zero-order hold or ZOH and has the transfer function $ZOH(s)$. We can compute $ZOH(s)$ by determining its impulse response. The hold filter will receive one unit-size impulse if the input signal is zero at every sample time except $t = 0$ and is equal to one there. In that case, $r^*(t) = \delta(t)$ and $r_h(t)$, which is now the impulse response of ZOH, is a pulse of height 1 and duration T seconds. The mathematical representation of the impulse response is, using the unit step function,

$$p(t) = 1(t) - 1(t - T).$$

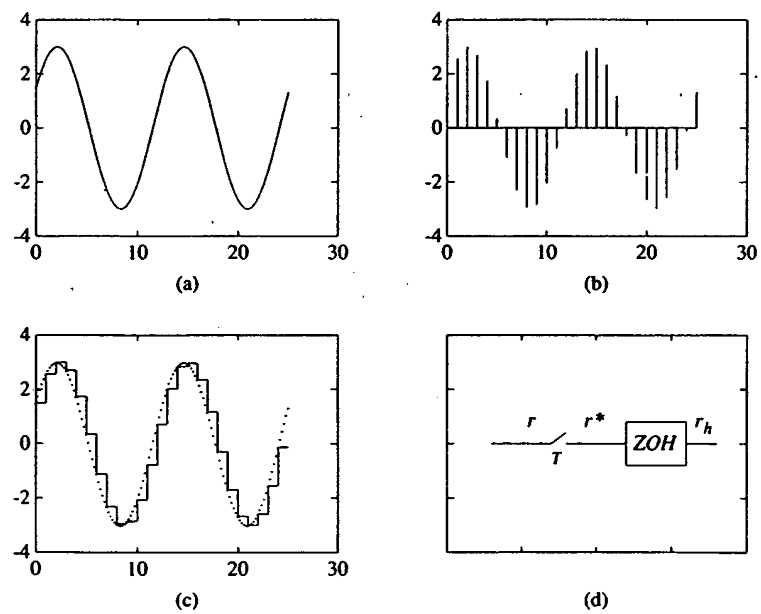

Figure 3.3 The sample and hold, showing typical signals. (a) Input signal r; (b) sampled signal r^*; (c) output signal r_h; (d) sample and hold.

The required transfer function is the Laplace transform of $p(t)$ as

$$\begin{aligned} ZOH(s) &= \mathcal{L}\{p(t)\} \\ &= \int_0^{\infty} [1(t) - 1(t-T)]e^{-st}dt \\ &= (1 - e^{-sT})/s. \end{aligned} \tag{3.7}$$

Thus the linear behavior of an A/D converter with sample and hold can be modeled by Fig. 3.3. We must emphasize that the impulsive signal $r^*(t)$ in Fig. 3.3 is not expected to represent a physical signal in the A/D converter circuit; rather it is a hypothetical signal introduced to allow us to obtain a transfer-function model of the hold operation and to give an input–output model of the sample-and-hold suitable for transform and other linear systems analysis.

3.3 SPECTRUM OF A SAMPLED SIGNAL AND ALIASING

We can get further insight into the process of sampling by an alternative representation of the transform of $r^*(t)$, using Fourier analysis. From (3.1) we see that $r^*(t)$ is a product of $r(t)$ and the train of impulses, $\Sigma\delta(t - kT)$. The latter series, being periodic, can be represented by a Fourier series,

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{n=-\infty}^{\infty} C_n e^{j(2\pi n/T)t},$$

where the Fourier coefficients, C_n, are given by the integral over one period as

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - kT)e^{-jn(2\pi t/T)}dt.$$

The only term in the sum of impulses that is in the range of the integral is the one at the origin $\delta(t)$, so the integral reduces to

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jn(2\pi t/T)}dt;$$

but the sifting property from (3.2) makes this easy to integrate, with the result

$$C_n = \frac{1}{T}.$$

Thus we have derived the representation for the sum of impulses as a Fourier series:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi n/T)t}. \tag{3.8}$$

We define $\omega_s = 2\pi/T$ as the radian sampling frequency and now substitute (3.8) into (3.1) using ω_s, and we take the Laplace transform of the output of the mathematical sampler,

$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r(t) \left\{ \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \right\} e^{-st} dt;$$

we integrate the sum, term by term

$$R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{jn\omega_s t} e^{-st} dt;$$

and if we combine the exponentials,

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{-(s-jn\omega_s)t} dt.$$

The integral here is the Laplace transform of $r(t)$ with only a change of variable where the frequency goes. The result can therefore be written as

$$R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} R(s - jn\omega_s), \tag{3.9}$$

where $R(s)$ is the transform of $r(t)$. In communication or radio engineering terms, (3.8) expresses the fact that the impulse train corresponds to an infinite sequence of carrier frequencies at integral values of $2\pi/T$, and (3.9) shows that when $r(t)$ modulates all these carriers, it produces a never-ending train of sidebands. A sketch of the elements in the sum given in (3.9) is shown in Fig. 3.4.

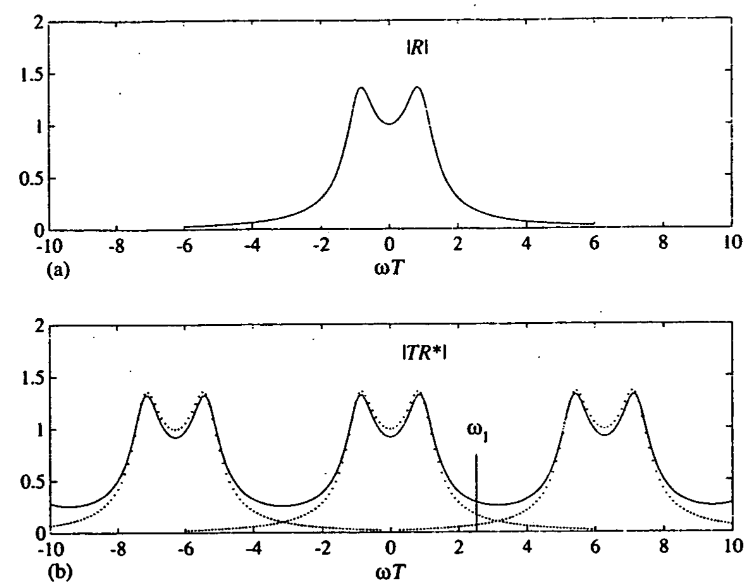

Figure 3.4 (a) Sketch of a spectrum amplitude and (b) the components of the spectrum after sampling, showing aliasing.

An important feature of sampling, shown in Fig. 3.4, is illustrated at the frequency marked ω_1. Two curves are drawn comprising two of the elements of the sum at ω_1. One of these, the larger amplitude in the figure located at the frequency ω_1, is the value of $R(j\omega_1)$. The smaller component at ω_1 comes from the spectrum centered at $2\pi/T$ and is $R(j\omega_0)$, where ω_0 is such that $\omega_0 = \omega_1 - 2\pi/T$. This frequency, ω_0, which shows up at ω_1 after sampling, is called in the trade an "alias" of ω_1; the phenomenon is called *aliasing*.

The phenomenon of aliasing has a clear meaning in time. Two continuous sinusoids of different frequencies appear at the same frequency when sampled. We cannot, therefore, distinguish between them based on their samples alone. Fig. 3.5 shows a plot of a sinusoid at $\frac{1}{8}$ Hz and of a sinusoid at $\frac{7}{8}$ Hz. If we sample these waves at 1 Hz, as indicated by the dots, then we get the same sample values from both signals and would continue to get the same sample values for all time. Note that the sampling frequency is 1, and, if $f_1 = \frac{1}{8}$, then

$$f_0 = \tfrac{1}{8} - 1 = -\tfrac{7}{8}.$$

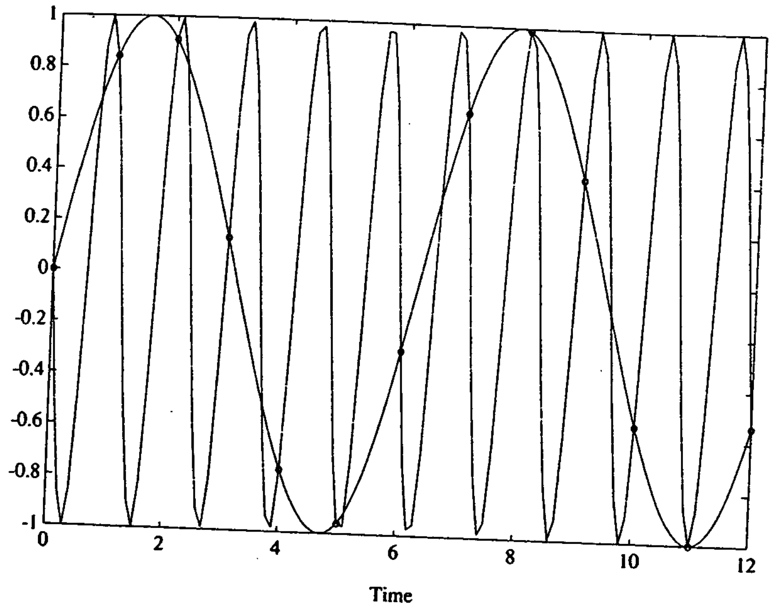

Figure 3.5 Plot of two sinusoids that have identical values at unit sampling intervals—an example of aliasing.

The significance of the negative frequency is that the $\frac{7}{8}$-Hz sinusoid in Fig. 3.5 is a negative sine function.

Thus, as a direct result of the sampling operation, when data are sampled at frequency $2\pi/T$ the total harmonic content that shows up at a given frequency ω_1 is to be found not only from the original signal at ω_1 but also from all those frequencies that are aliases of ω_1, namely, components from all frequencies $\omega_1 + n2\pi/T = \omega_1 + n\omega_s$ as shown in the formula of (3.9) and sketched in Fig. 3.4. The errors caused by this aliasing can be very severe, especially if a substantial amplitude of high-frequency noise is present with the signal to be sampled. To minimize the error caused by this effect, it is standard practice to *precede* the sampling operation (such as the sample-and-hold circuit discussed earlier) by a low-pass antialias filter that will remove (almost) all spectral content above the half-sampling frequency, i.e., above π/T. A sketch of the result is drawn in Fig. 3.6.

If all spectral content above the frequency π/T is removed, then no aliasing is introduced by sampling and the signal spectrum is undistorted, even though it is repeated endlessly, centered at $n2\pi/T$. The critical frequency, π/T, was first reported by H. Nyquist, and is called the Nyquist frequency.

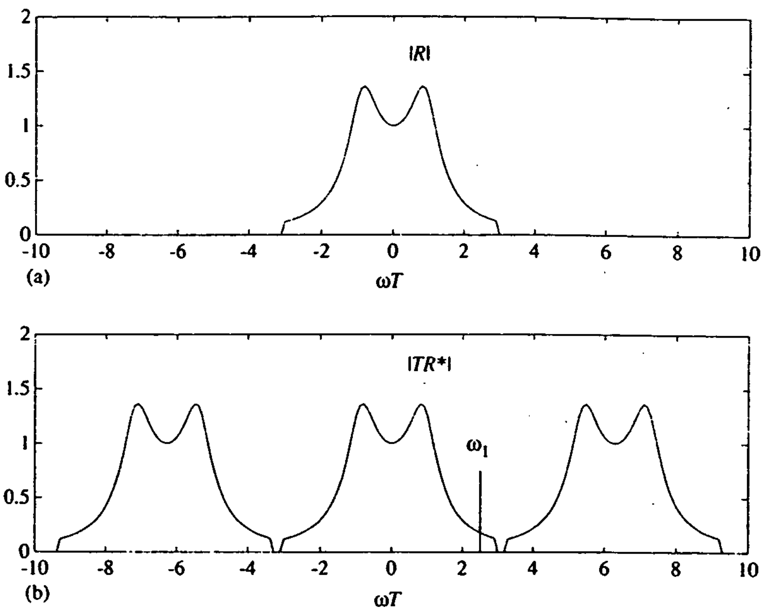

Figure 3.6 (a) Sketch of a spectrum amplitude and (b) the components of the spectrum after sampling, showing removal of aliasing with an antialiasing filter.

Band-limited signals that have no components above the Nyquist frequency are represented unambiguously by their samples.

A corollary to the aliasing issue is the sampling theorem. We have seen that if $R(j\omega)$ has components above the Nyquist frequency $\omega_s/2$ or π/T, then overlap and aliasing will occur. Conversely, we noticed that if $R(j\omega)$ is zero for $|\omega| \geq \pi/T$, then sampling at intervals of T sec will produce no aliasing and the original spectrum can be recovered exactly from R^*, the spectrum of the samples. Once the spectrum is recovered, by inverse transform we can calculate the original signal itself. This is the sampling theorem: One can recover a signal from its samples, if the sampling frequency ($\omega_s = 2\pi/T$) is *at least twice* the highest frequency (π/T) in the signal. Notice that the sampling theorem requires that $R(j\omega)$ is exactly zero for all frequencies above π/T.

A phenomenon somewhat related to aliasing is that of ***hidden oscillations.*** If signal frequencies only up to π/T can be sampled without confusion, there is the possibility that a signal could contain some frequencies that the samples do not show *at all*. Such signals, when they occur in a digital control, are called "hidden oscillations," an example of which is shown in Fig. 5.29.

3.4 DATA EXTRAPOLATION AND IMPOSTORS

The sampling theorem states that under the right conditions, it is possible to recover a signal from its samples; we now consider a formula for doing so. From Fig. 3.6 we can see that the spectrum of $R(j\omega)$ is contained in the low-frequency part of $R^*(j\omega)$. Therefore, to recover $R(j\omega)$ we need only process $R^*(j\omega)$ through a low-pass filter and multiply by T to regain R. As a matter of fact, if $R(j\omega)$ has zero energy for frequencies in the bands above π/T (such an R is said to be band-limited), then an *ideal low-pass filter* with gain T for $-\pi/T \leq \omega \leq \pi/T$ and zero elsewhere would recover $R(j\omega)$ from $R^*(j\omega)$ exactly. Suppose we define this ideal low-pass filter characteristic as $L(j\omega)$. Then we have the result

$$R(j\omega) = L(j\omega)R^*(j\omega). \tag{3.10}$$

The signal $r(t)$ is the inverse transform of $R(j\omega)$, and because by (3.10) $R(j\omega)$ is the *product* of two transforms, its inverse transform $r(t)$ must be the convolution of the time functions $\ell(t)$ and $r^*(t)$. The form of the filter impulse response can be computed by using the definition of $L(j\omega)$ from which the inverse transform gives[3]

$$\begin{aligned}
\ell(t) &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} T e^{j\omega t} d\omega \\
&= \frac{T}{2\pi}\frac{e^{j\omega t}}{jt}\bigg|_{-\pi/T}^{\pi/T} \\
&= \frac{T}{2\pi j t}\left(e^{j(\pi t/T)} - e^{-j(\pi t/T)}\right) \\
&= \frac{1}{\pi t/T}\sin\frac{\pi t}{T} \\
&= \operatorname{sinc}\frac{\pi t}{T}.
\end{aligned} \tag{3.11}$$

Using (3.1) for $r^*(t)$ and (3.11) for $\ell(t)$, we find that their convolution is

$$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT)\operatorname{sinc}\frac{\pi(t-\tau)}{T} d\tau.$$

[3] "Sinc" is the name given to the function defined by $\operatorname{sinc}(\theta) = \sin(\theta)/\theta$.

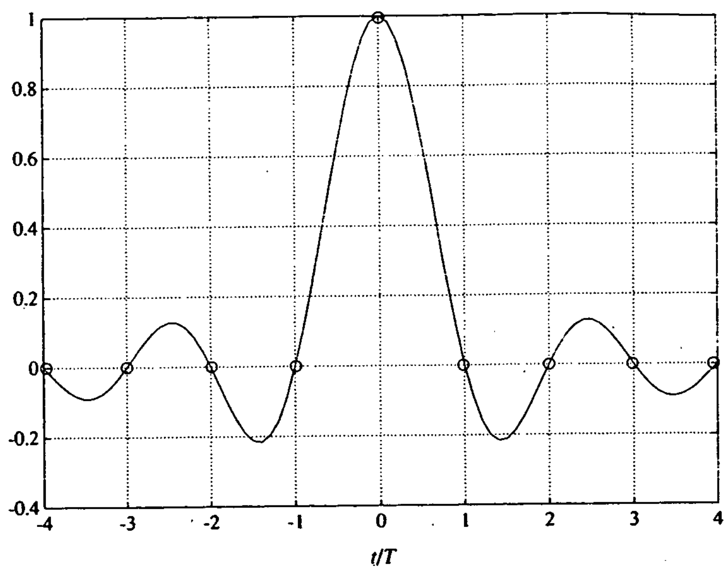

Figure 3.7 Plot of the impulse response of the ideal low-pass filter.

Using the sifting property of the impulse, we have

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT)\mathrm{sinc}\frac{\pi(t-kT)}{T}. \tag{3.12}$$

Equation (3.12) is a constructive statement of the sampling theorem: It shows explicitly how to construct the (by assumption) band-limited function $r(t)$ from its samples. The sinc functions are the interpolators that fill in the time gaps between samples with a wave that has no frequencies above π/T. A plot of the impulse response of this "ideal" hold filter is drawn in Fig. 3.7 from the formula of (3.11).

There is one serious drawback to the extrapolating signal given by (3.11). Because $\ell(t)$ is the impulse response of the ideal low-pass filter $L(j\omega)$, it follows that this filter is noncausal because $\ell(t)$ is nonzero for $t < 0$. $\ell(t)$ starts at $t = -\infty$ when the impulse that triggers it does not occur until $t = 0$! In many communications problems the interpolated signal is not needed until well after the samples are acquired, and the noncausality can be overcome by adding a phase lag, $e^{-j\omega\lambda}$, to $L(j\omega)$, which adds a *delay* to the filter and to the signals processed through it. In feedback control systems, a large delay is usually disastrous for stability, so we avoid such approximations to this function and use something else, like the polynomial

holds, of which the zero-order hold already mentioned in connection with the ADC is the most elementary.

In Section 3.2 we introduced the zero-order hold as a model for the storage register in an A/D converter that maintains a constant signal value between samples. We showed in (3.7) that it has the transfer function

$$ZOH(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}. \tag{3.13}$$

We can discover the frequency properties of ZOH by expressing (3.13) in magnitude and phase form. To do this, we factor out $e^{-j\omega T/2}$ and multiply and divide by $2j$ to write the transfer function in the form

$$ZOH(j\omega) = e^{-j\omega T/2}\left\{\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right\}\frac{2j}{j\omega}.$$

The term in brackets is recognized as the sine, so this can be written

$$ZOH(j\omega) = Te^{-j\omega T/2}\frac{\sin(\omega T/2)}{\omega T/2}$$

and, using the definition of the sinc function,

$$ZOH(j\omega) = e^{-j\omega T/2}T\operatorname{sinc}(\omega T/2). \tag{3.14}$$

Thus the effect of the zero-order hold is to introduce a phase shift of $\omega T/2$, corresponding to a time delay of $T/2$ seconds, and to multiply the gain by a function with the magnitude of $\operatorname{sinc}(\omega T/2)$. A plot of the magnitude is shown in Fig. 3.8, which illustrates the fact that although the zero-order hold is a low-pass filter, it has a cut-off frequency well beyond the Nyquist frequency. The magnitude function is

$$|ZOH(j\omega)| = T\left|\operatorname{sinc}\frac{\omega T}{2}\right|, \tag{3.15}$$

which slowly gets smaller as ω gets larger until it is zero for the first time at $\omega = \omega_s = 2\pi/T$. The phase is

$$\angle ZOH(j\omega) = \frac{-\omega T}{2}, \tag{3.16}$$

plus the 180° shifts where the sinc function changes sign.

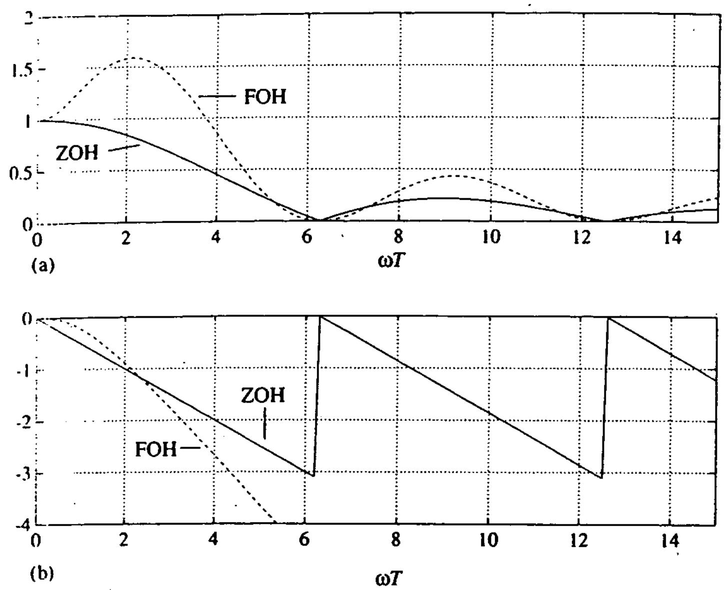

Figure 3.8 (a) Magnitude and (b) phase of polynomial hold filters.

We can now give a complete analysis of the sample-and-hold circuit of Fig. 3.3 for a sinusoidal input $r(t)$ in both the time and the frequency domains. We consider first the time domain, which is simpler, being just an exercise in construction. For purposes of illustration, we will use $r(t) = 3\sin(50t + \pi/6)$ as plotted in Fig. 3.9.

If we sample $r(t)$ at the instants kT where the sampling frequency is $\omega_s = 2\pi/T = 200\pi$ and $T = 0.01$, then the plot of the resulting $r_h(kT)$ is as shown in Fig. 3.9. Notice that although the input is a single sinusoid, the output is clearly *not* sinusoidal. Thus it is not possible to describe this system by a transfer function, because the fundamental property of linear, constant systems is that a sinusoid input produces an output that is a sinusoid of the same frequency, and the complex ratio of the amplitudes and phases is the transfer function. The sample-and-hold system is linear but time varying. In the frequency domain, it is clear that the output $r_h(t)$ contains more than one frequency, and a complete analysis requires that we compute the amplitudes and phases of them all. However, in the application to control systems, the output of the hold will typically be applied to a dynamical system that is of low-pass character; thus the most important component in $r_h(t)$ is the fundamental harmonic, at $\omega_o = 50$ rad/sec in this case. Also, in the important field of digital filtering, one is usually using the digital filter to

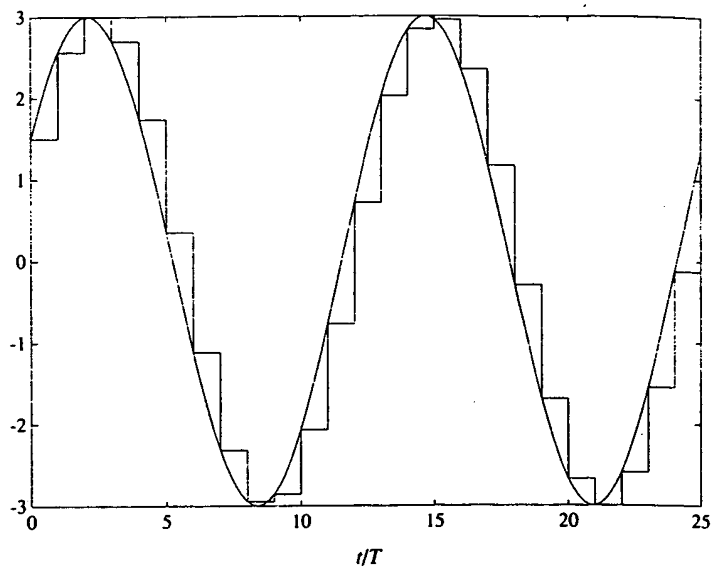

Figure 3.9 Plot of $3\sin(50t+\pi/3)$ and the output of a sample-and-hold with sample period $T = 0.01$.

replace an analog filter, and it is the fundamental that represents the signal component in the output. The other harmonics that appear in the output are *impostors*, posing as signal when they are really unwanted consequences of the sample-and-hold process. In any event, we can proceed to analyze $r_h(t)$ for all its harmonics and select out the fundamental component when it makes sense, either by analysis or, in the implementation, by a low-pass anti-impostor filter.

First, we need the spectrum of $r(t)$. Because a sinusoid can be decomposed into two exponentials, we consider the Fourier transform of $v(t) = e^{j\omega_o t + j\phi}$. For this we have

$$V(j\omega) = \int_{-\infty}^{\infty} e^{(j\omega_o t + j\phi)} e^{-j\omega t} dt. \tag{3.17}$$

This integral does not converge in any obvious way, but we can approach it from the back door, as it were. Consider again the impulse, $\delta(t)$. The direct transform of this object is easy, considering the sifting property, as follows:

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1.$$

Now the general form of the inverse Fourier transform is given by the expression

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega.$$

If we apply the inverse transform integral to the impulse and its transform, we take $f(t) = \delta$ and $F(j\omega) = 1$ with the result

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}d\omega.$$

However, except for notation and a simple change of variables, this is exactly the integral we needed to evaluate the spectrum of the single exponential. If we exchange t with ω the integral reads

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}dt.$$

Equation 3.17 is of this form

$$\begin{aligned} V(j\omega) &= \int_{-\infty}^{\infty} e^{(j\omega_o t + j\phi)}e^{-j\omega t}dt \\ &= e^{j\phi}\int_{-\infty}^{\infty} e^{jt(\omega_o - \omega)}dt \\ &= 2\pi e^{j\phi}\delta(\omega - \omega_o). \end{aligned} \tag{3.18}$$

At the last step in this development, the sign of the argument in the delta function was changed, because δ is an even function and the order is more natural as $(\omega - \omega_o)$ rather than the opposite. We can now express the spectrum of $r(t) = A\cos(\omega_o t + \phi)$ in terms of impulse functions. It is

$$R(j\omega) = \pi A[e^{j\phi}\delta(\omega - \omega_o) + e^{-j\phi}\delta(\omega + \omega_o)]$$

and consists of two impulses at ω_o and $-\omega_o$ of intensity πA and phase ϕ and $-\phi$, respectively. A sketch of this spectrum is shown in Fig 3.10(a) for $A = 1/\pi$. We represent the impulses by arrows whose heights are proportional to the intensities of the impulses.

After sampling, as we saw in (3.9), the spectrum of R^* is directly derived from that of R as the sum of multiple copies of that of R shifted by $n2\pi/T$ for all integers n and multiplied by $1/T$. A plot of the result is shown in Fig.

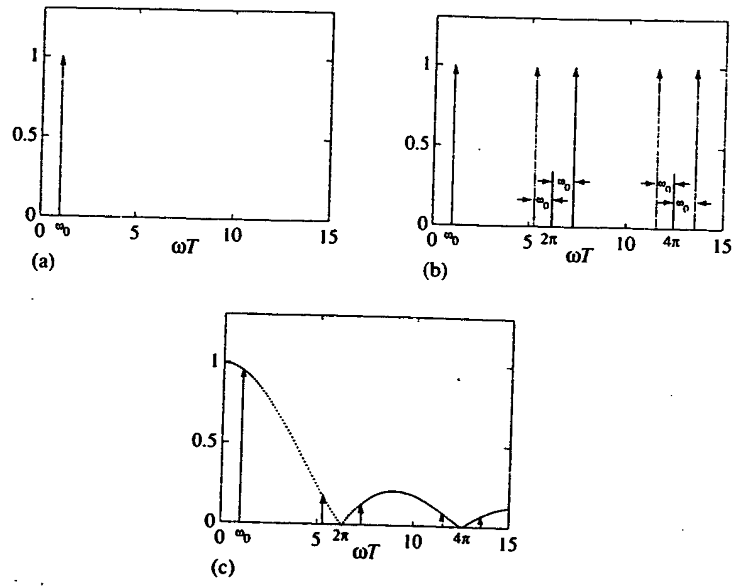

Figure 3.10 Plot of the spectra of (a)R, (b)R^*, and (c)R_h.

3.10(b) normalized by T. Finally, to find the spectrum of R_h, we need only multiply the spectrum of R^* by the transfer function of the ZOH, which is

$$ZOH(j\omega) = Te^{-j\omega T/2}\mathrm{sinc}(\omega T/2).$$

Thus the spectrum of R_h is also a sum of an infinite number of terms, but now with intensities modified by the sinc function and phases shifted by the delay function $\omega T/2$. These intensities are plotted in Fig. 3.10(c). Naturally, when all the harmonics included in R_h are converted to their time functions and added, they sum to the piecewise, constant, staircase function plotted earlier in Fig. 3.9.

If we want a best approximation to r_h using only one sinusoid, we need only take out the first or fundamental harmonic from the components of R^*. This component has phase shift ϕ and amplitude A sinc$(\omega T/2)$. In the time domain, the corresponding sinusoid is given by

$$v_1(t) = A[\mathrm{sinc}(\omega T/2)]\sin[\omega_o(t - \tfrac{T}{2})].$$

A plot of this approximation for the signal from Fig 3.9 is given in Fig. 3.11 along with both the original input and the sampled-and-held output to show the nature of the approximation.

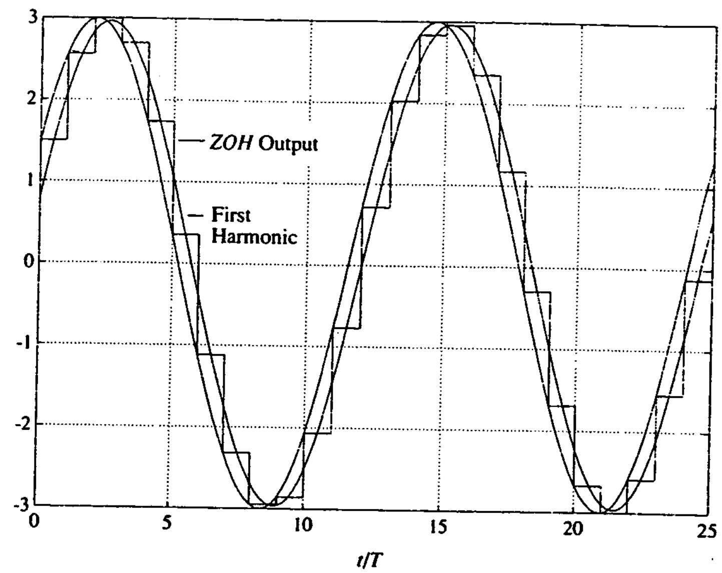

Figure 3.11 Plot of the output of the sample and hold and the first harmonic approximation.

In control design, we can frequently achieve a satisfactory design for a sampled-data system by approximating the sample and hold with a continuous transfer function corresponding to the delay of $T/2$. The controller design is then done in the continuous domain but is implemented by computing a discrete equivalent. More discussion of this technique, sometimes called *emulation*, will be given in Chapter 5, where some examples illustrate the results.

First-Order Hold. A more complex circuit that might be preferred to the zero-order hold is the first-order hold, which, as suggested by its name, extrapolates data between sampling periods by a first-order polynomial, a straight line. We can, as with the zero-order hold, compute the transfer function of the filter that, acting with impulse sampling, produces the action of a first-order hold. A sketch of the response to a unit pulse will be helpful and is shown in Fig. 3.12. The first line, rising from 1 to 2 over the period from $t = 0$ to $t = T$, is an extrapolation of the line between the points at $t = -T$ (where the sample was, by assumption, equal to zero) and at $t = 0$, where the sample was unity. Likewise, the line going in the negative

direction from 0 at $t = T$ to -1 at $t = 2T$ is the extrapolation of the line between the points at $t = 0$ and $t = T$. The Laplace transform of this $h(t)$ is

$$FOH(s) = T\left(\frac{1 - e^{-Ts}}{Ts}\right)^2 (Ts + 1). \tag{3.19}$$

The magnitude and phase of $FOH(j\omega)$ were plotted in Fig. 3.8 to permit comparison with the characteristics of ZOH. Note that for low frequencies (below $\pi/2T$) the first-order hold has significantly less phase lag than does the zero-order hold. However, the FOH has much more amplitude distortion than the ZOH does. No clear guidelines seem to exist indicating that one is preferred to the other from the standpoint of an ideal control-system response; however, the increased hardware complexity of the FOH implementation almost always dictates that the ZOH be used. In some cases (e.g., hydraulic systems), the steps from the ZOH have been found to be detrimental to the control actuator, and the solution has been to simply add a low-pass filter to the ZOH output with a time constant on the order of the sample period. Following our earlier definition, this would be called an anti-imposter filter; some prefer the term "smoothing filter." The filter is to be considered part of the plant, and its effects are taken into account in the design.

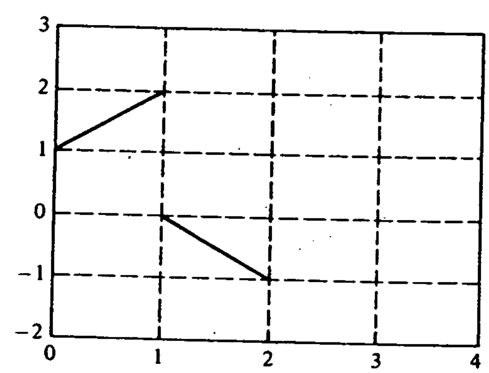

Figure 3.12 Impulse response of first-order-hold filter.

Obviously many other, more sophisticated, data extrapolators can and will be designed. However, rarely is additional complexity justified in terms of improved performance in feedback control.

3.5 BLOCK-DIAGRAM ANALYSIS

We have thus far talked mainly about discrete, continuous, and sampled signals. To analyze a feedback system that contains a digital computer, we need to be able to compute the transforms of output signals of systems that contain sampling operations in various places, including feedback loops, in the block diagram. The technique for doing this is a simple extension of the ideas of block-diagram analysis of systems that are all continuous or all discrete, but one or two rules need to be carefully observed to assure success. First, we should review the facts of sampled-signal analysis.

We represent the process of sampling a continuous signal and holding it by impulse modulation followed by low-pass filtering. For example, the system of Fig. 3.13 leads to

$$
\begin{aligned}
E(s) &= R^*(s)H(s),\\
U(s) &= E^*(s)G(s).
\end{aligned}
\tag{3.20}
$$

The result of impulse modulation of continuous-time signals like $e(t)$ and $u(t)$ is to produce a series of sidebands as given in (3.9) and plotted in Fig. 3.4, which result in periodic functions of frequency. If the transform of the signal to be sampled is a product of a transform that is already periodic of period $2\pi/T$, and one that is not, as in $U(s) = E^*(s)G(s)$, where $E^*(s)$ is periodic and $G(s)$ is not, we can show that $E^*(s)$ comes out as a factor of the result. This is the most important relation for the block-diagram analysis of sampled-data systems, namely,[4]

$$
U^*(s) = (E^*(s)G(s))^* = E^*(s)G^*(s). \tag{3.21}
$$

We can prove (3.21) either in the frequency domain, using (3.9), or in the time domain, using (3.1) and convolution. We will use (3.9) here. If $U(s) =$

[4] We of course assume the existence of $U^*(s)$, which is assured if $G(s)$ tends to zero as s tends to infinity at least as fast as $1/s$. We must be careful to avoid impulse modulation of impulses, for $\delta(t)\delta(t)$ is undefined.

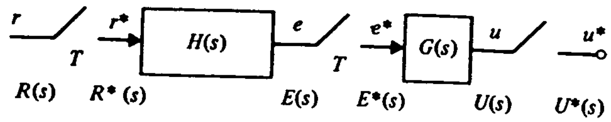

Figure 3.13 A cascade of samplers and filters.

$E^*(s)G(s)$, then by definition we have

$$U^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E^*(s - jn\omega_s)G(s - jn\omega_s); \tag{3.22}$$

but $E^*(s)$ is

$$E^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s),$$

so that

$$E^*(s - jn\omega_s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s - jn\omega_s). \tag{3.23}$$

Now in (3.23) we can let $k = \ell - n$ to get

$$\begin{aligned} E^*(s - jn\omega_s) &= \frac{1}{T} \sum_{\ell=-\infty}^{\infty} E(s - j\ell\omega_s) \\ &= E^*(s). \end{aligned} \tag{3.24}$$

In other words, because E^* is already periodic, shifting it an integral number of periods leaves it unchanged. Substituting (3.24) into (3.22) yields

$$\begin{aligned} U^*(s) &= E^*(s) \frac{1}{T} \sum_{-\infty}^{\infty} G(s - jn\omega_s) \\ &= E^*(s)G^*(s). \qquad QED \end{aligned} \tag{3.25}$$

Note especially what is *not* true. If $U(s) = E(s)G(s)$, then $U^*(s) \neq E^*(s)G^*(s)$ but rather $U^*(s) = (EG)^*(s)$. The periodic character of E^* in (3.21) is crucial.

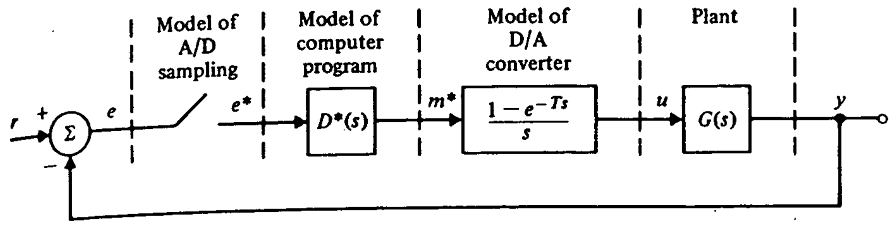

Figure 3.14 Block diagram of digital control as a sampled-data system.

The final result we require is that, given a sampled-signal transform such as $U^*(s)$, we can find the corresponding z-transform simply by letting $e^{sT} = z$ or

$$U(z) = U^*(s)\,|_{e^{sT}=z}\,. \tag{3.26}$$

There is an important time-domain reflection of (3.26). The inverse Laplace transform of $U^*(s)$ is the sequence of *impulses* with intensities given by $u(kT)$; the inverse z-transform of $U(z)$ is the sequence of values $u(kT)$. Conceptually, sequences of values and the corresponding z-transforms are easy to think about as being processed by a computer program, whereas the model of sampling as a sequence of impulses is what allows us to analyze a discrete system embedded in a continuous world. Of course, the impulse modulator must *always* be followed by a low-pass circuit (hold circuit) in the physical world. Note that (3.26) can also be used in the other direction to obtain $U^*(s)$, the Laplace transform of the train of impulses, from a given $U(z)$.

These rules of analysis can be illustrated by example. Consider the block diagram given in Fig. 3.14 taken from Fig. 1.1. In Fig. 3.14 we have modeled the A/D converter plus computer program plus D/A converter as an impulse modulator [which takes the samples from $e(t)$], a computer program that processes these samples, and a zero-order hold that constructs the piecewise, constant output of the D/A converter from the impulses of m^*. In the actual computer we assume that the samples of $e(t)$ are manipulated by a difference equation whose input–output effect is described by the z-transform $D(z)$. These operations are represented in Fig. 3.14 *as if* they were performed on impulses, and hence the transfer function is $D^*(s)$ according to (3.26). Finally, the manipulated impulses, $m^*(t)$, are applied to the zero-order hold from which the piecewise-constant-control signal $u(t)$ comes. In reality, of course, the computer operates on the sample values of $e(t)$ and the piecewise-constant output is generated via a storage register and a D/A converter. The

impulses provide us with a convenient, consistent, and effective model of the processes to which Laplace-transform methods can be applied.

From the results given thus far, we can write relations among Laplace transforms as

$$E(s) = R - Y, \tag{3.27}$$

$$M^*(s) = E^*D^*, \tag{3.28}$$

$$U = M^* \left[\frac{1 - e^{Ts}}{s}\right], \tag{3.29}$$

$$Y = GU. \tag{3.30}$$

The usual idea is to relate the discrete output, Y^*, to the discrete input, R^*. Suppose we sample each of these equations by using the results of (3.5) to "star" each transform. The equations are[5]

$$E^* = R^* - Y^*, \tag{3.31}$$

$$M^* = E^*D^*, \tag{3.32}$$

$$U^* = M^*, \tag{3.33}$$

$$Y^* = [GU]^*. \tag{3.34}$$

Now (3.34) indicates that we need U, not U^*, to compute Y^*, so we must back up to substitute (3.29) into (3.34):

$$Y^* = \left[GM^*\left(\frac{1 - e^{Ts}}{s}\right)\right]^*. \tag{3.35}$$

Taking out the periodic parts, which are those in which s appears only as e^{sT} [which include $M^*(s)$], we have

$$Y^* = (1 - e^{-Ts})M^*\left(\frac{G}{s}\right)^*. \tag{3.36}$$

Substituting from (3.28) for M^* gives

$$Y^* = (1 - e^{-Ts})E^*D^*(G/s)^*. \tag{3.37}$$

[5]In sampling (3.29) we obtain (3.33) by use of the convention given in footnote 2, which follows (3.5) for impulse modulation of discontinuous functions. From the time-domain operation of the zero-order hold, it is clear that the samples of u and m are the same, and then from this (3.33) follows.

And substituting (3.31) for E^* yields

$$Y^* = (1 - e^{-Ts})D^*(G/s)^*[R^* - Y^*]. \tag{3.38}$$

If we call

$$(1 - e^{-Ts})D^*(G/s)^* = H^*,$$

then we can solve (3.38) for Y^*, obtaining

$$Y^* = \frac{H^*}{1 + H^*}R^*. \tag{3.39}$$

Example 3.1: These equations can be illustrated with a simple example. Suppose that our plant has the first-order transfer function

$$G(s) = \frac{a}{s + a}, \tag{3.40}$$

that the computer program corresponds to a discrete integrator

$$u(kT) = u(kT - T) + K_0 e(kT), \tag{3.41}$$

and that the computer D/A holds the output constant so that the zero-order hold is the correct model. Suppose we select the sampling period T so that $e^{-aT} = \frac{1}{2}$. We wish to compute the components of H^* given in (3.39). For the computer program we have the transfer function of (3.41), which in terms of z is

$$D(z) = \frac{U(z)}{E(z)} = \frac{K_0}{1 - z^{-1}} = \frac{K_0 z}{z - 1}.$$

Using (3.26), we get the Laplace-transform form

$$D^*(s) = \frac{K_0 e^{sT}}{e^{sT} - 1}. \tag{3.42}$$

For the plant and zero-order-hold we require[6]

$$(1-e^{-Ts})(G(s)/s)^* = (1-e^{-Ts})\left(\frac{a}{s(s+a)}\right)^*$$
$$= (1-e^{-Ts})\left(\frac{1}{s}-\frac{1}{s+a}\right)^*.$$

Using (3.5), we have

$$(1-e^{-Ts})(G(s)/s)^* = (1-e^{-Ts})\left(\frac{1}{1-e^{-Ts}}-\frac{1}{1-e^{-aT}e^{-Ts}}\right).$$

Because we assumed (for simplicity) that $e^{-aT}=\frac{1}{2}$, this reduces to

$$(1-e^{-Ts})(G(s)/s)^* = \frac{(1/2)e^{-Ts}}{1-(1/2)e^{-Ts}}$$
$$= \frac{1/2}{e^{Ts}-1/2}. \tag{3.43}$$

Combining (3.43) and (3.42), then, in this case, we obtain

$$H^*(s) = \frac{K_0}{2}\,\frac{e^{sT}}{(e^{sT}-1)(e^{sT}-1/2)}. \tag{3.44}$$

Equation (3.44) can now be used in (3.39) to find the closed-loop transfer function from which the dynamic and static responses can be studied, as a function of K_0, the program gain. We note also that beginning with (3.27), we can readily calculate that

$$Y(s) = R^*\frac{D^*}{1+H^*}\,\frac{(1-e^{-Ts})}{s}\,G(s). \tag{3.45}$$

Equation (3.45) shows how to compute the response of this system in between sampling instants. For a given $r(t)$, the starred terms in (3.45) and the $(1-e^{-Ts})$-term correspond to a train of impulses whose individual values can be computed by expanding in powers of e^{-Ts}. These impulses are applied to $G(s)/s$, which is the step response of the plant. Thus, between sampling instants, we will see segments of the plant step response.

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[6]Notice the similarity with (2.39) and Example 2.8.

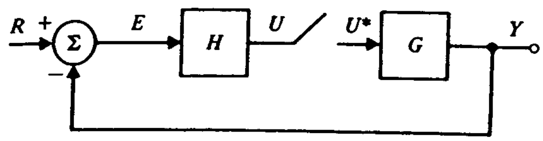

Figure 3.15 A simple system that does not have a transfer function.

With the exception of the odd-looking forward transfer function, (3.39) looks like the familiar feedback formula: forward-over-one-plus-feedback. Unfortunately, the sequence of equations by which (3.39) was computed was a bit haphazard, and such an effort might not always succeed. Another example will further illustrate the problem.

Example 3.2: Consider the block diagram of Fig. 3.15, which has only one sampling operation. This situation can arise if the error sensor has significant dynamics that precede the sampling action of the A/D converter. In this case $H(s)$ represents the sensor dynamics. Again, we write the equations (all symbols are Laplace transforms):

$$E = R - Y, \tag{3.46}$$

$$U = HE, \tag{3.47}$$

$$Y = U^*G; \tag{3.48}$$

and again we sample

$$E^* = R^* - Y^*, \tag{3.49}$$

$$U^* = (HE)^*, \tag{3.50}$$

$$Y^* = U^*G^*. \tag{3.51}$$

How do we solve? In (3.50) we need E, not E^*. So we must go back to (3.46):

$$\begin{aligned} U^* &= (H(R-Y))^* \\ &= (HR)^* - (HY)^*. \end{aligned}$$

Using (3.48) for Y, we have

$$U^* = (HR)^* - (HU^*G)^*.$$

Taking out the periodic U^* in the second term on the right gives

$$U^* = (HR)^* - U^*(HG)^*.$$

Solving, we get

$$U^* = \frac{(HR)^*}{1 + (HG)^*}. \tag{3.52}$$

From (3.45), we can solve for Y^*:

$$Y^* = \frac{(HR)^*}{1 + (HG)^*} G^*. \tag{3.53}$$

Equation (3.53) displays a curious fact. The transform of the input is bound up with $H(s)$ and *cannot* be divided out to give a transfer function! This system displays an important fact that all our facile manipulations of samples, and so on, might cause us to neglect: A sampled-data system is *time varying.* The response depends on the time *relative to the sampling instant* at which the signal is applied. Only when the input samples *alone* are required to generate the output samples can we obtain a discrete transfer function. The time variation occurs on the taking of samples. In general, as in Fig. 3.15, the entire input signal $r(t)$ is involved in the system response, and the transfer-function concept fails. Even in the absence of a transfer function, however, the techniques developed here permit study of stability and response to specific inputs such as step, ramp, or sinusoidal signals.

We need to know the general rules of block-diagram analysis. In solving Fig. 3.15 we found ourselves working with U, the signal that was sampled. This is in fact the key to the problem. Given a block diagram with several samplers, *always select the variables at the inputs to the samplers as the unknowns.* Being sampled, these variables have periodic transforms and will always "come free" after the equation sampling process and give a set of starred variables for which we can solve.

Example 3.3: Consider as a final example the block diagram drawn in Fig. 3.16. We select E and M as independent variables

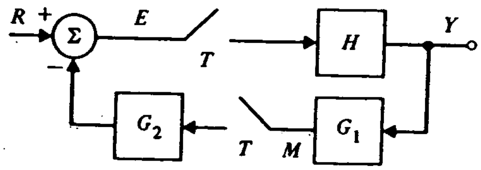

Figure 3.16 A final example for transfer-function analysis of sampled-data systems.

and write

$$E(s) = R - M^* G_2, \tag{3.54}$$

$$M(s) = E^* H G_1. \tag{3.55}$$

Next we sample these signals, and use the "if periodic, then out" rule from (3.21):

$$E^* = R^* - M^* G_2^*, \tag{3.56}$$

$$M^* = E^* (HG_1)^*. \tag{3.57}$$

We solve these equations by substituting for M^* in (3.56) from (3.57):

$$\begin{aligned} E^* &= R^* - E^*(HG_1)^* G_2^* \\ &= \frac{R^*}{1 + (HG_1)^* G_2^*}. \end{aligned} \tag{3.58}$$

To obtain Y we use the equation

$$\begin{aligned} Y &= E^* H \\ &= \frac{R^* H}{1 + (HG_1)^* G_2^*}, \end{aligned} \tag{3.59}$$

and

$$Y^* = \frac{R^* H^*}{1 + (HG_1)^* G_2^*}. \tag{3.60}$$

In this case we have a transfer function. Why? Because only the samples of the external input are used to cause the output. To obtain

the z-transform of the samples of the output, we would let $e^{sT} = z$ in (3.60). From (3.59) we can solve for the continuous output, which consists of impulses applied to $H(s)$ in this case.

3.6 SUMMARY

In this chapter we have considered the analysis of mixed systems that are partly discrete and partly continuous, taking the continuous point of view. We used impulse modulation to represent the sampling process, and we derived the transfer functions of filters that would represent zero-order and first-order hold action. We showed that the transform of a sampled signal is periodic and that sampling introduces aliasing, which may be interpreted in both the frequency and the time domains. From the condition of no aliasing we derived the sampling theorem.

Finally, we presented the block-diagram analysis of sampled-data systems, showing that proper techniques, including the treatment of the sampler inputs as unknowns, would lead to solution for the output transforms. However, we also found that not every sampled-data system has a transfer function.

PROBLEMS AND EXERCISES

3.1 Derive (3.45).

3.2 Sketch a signal that shows hidden oscillations.

3.3 Consider the circuit of Fig. 3.17. By plotting the response to a signal that is zero for all sample instants except $t = 0$ and that is 1.0 at $t = 0$, show that this circuit implements a first-order hold.

3.4 Sketch the step response $y(t)$ of the system shown in Fig. 3.18 for $k = \frac{1}{2}$, 1, and 2.

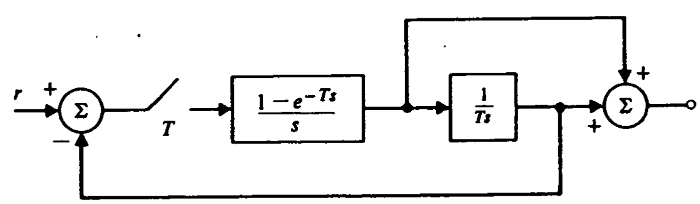

Figure 3.17 Block diagram of a first-order hold.

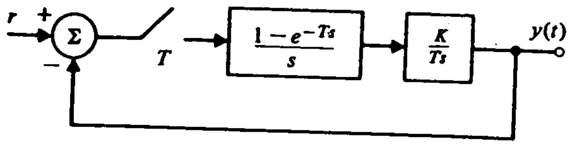

Figure 3.18 A sampled-data system.

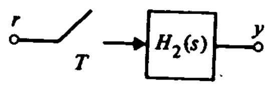

Figure 3.19 A general hold circuit.

3.5 Sketch the response of a *second*-order hold circuit to a step unit. What might be the major disadvantage of this data extrapolator? See Fig. 3.19.

3.6 Find the transform of the output $Y(s)$ and its samples $Y^*(s)$ for the block diagrams shown in Fig. 3.20. Indicate whether a transfer function exists in each case.

3.7 Assume the following transfer functions are preceded by a zero-order hold and compute the resulting discrete transfer functions.

a) $G_1(s) = 1/s^2$

b) $G_2(s) = e^{-1.5s}/(s+1)$

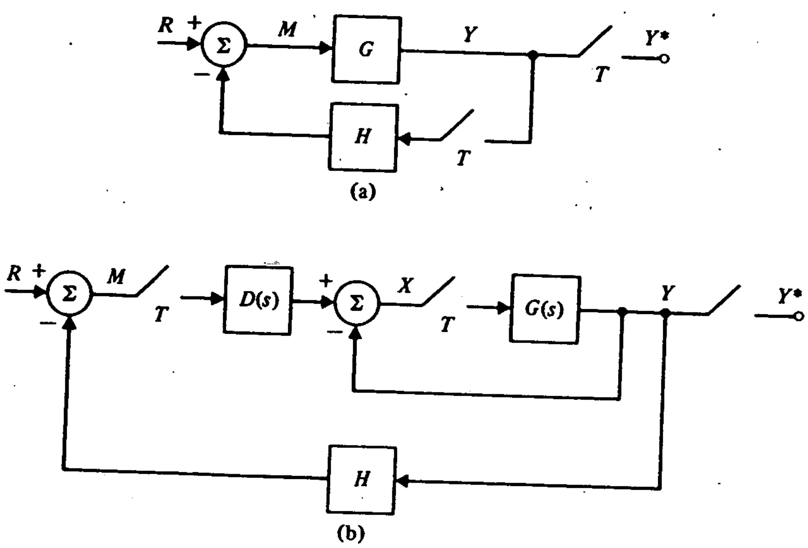

Figure 3.20 Block diagrams of sampled data systems. (a) Single loop; (b) multiple loop.

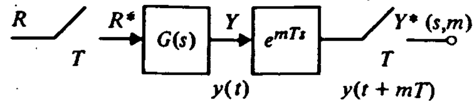

Figure 3.21 Block diagrams showing the modified z-transform.

c) $G_3(s) = 1/s(s+1)$
d) $G_7(s) = e^{-1.5s}/s(s+1)$
e) $G_8(s) = 1/(s^2-1)$

3.8 One technique for examining the response of a sampled-data system between sampling instants is to shift the response a fraction of a sample period to the left and sample the result. The effect is as shown in the block diagram of Fig. 3.21 and is described by the equation

$$Y^*(s,m) = R^*(s)\mathcal{Z}\{G(s)e^{mTs}\}.$$

The function $\mathcal{Z}\{G(s)e^{mTs}\}$ is called the *modified z-transform* of $G(s)$. Let

$$G(s) = \frac{1}{s+1}, \quad T = 1, \quad R(s) = \frac{1}{s}.$$

a) Compute $y(t)$ by constructing the samples $y(kT)$ from $Y^*(s)$ and observing that with this plant, $y(t)$ is an exponential decay with unit time constant over the intersample interval. Sketch the response for five sample intervals.

b) Let $m = \frac{1}{2}$ and compute the samples corresponding to $Y^*(s;m)$ [or $Y(z;m)$]. Plot these on the same sketch as the samples of part (a) and verify that the midway points have been found.

3.9 Show how to construct a signal of "hidden oscillations," even one that grows in an unstable fashion but whose sample values are zero. Where in the s-plane are the poles of the transforms of your signal(s)?